Kinematics and Dynamics of Machines

3. Kinematic Analysis of Mechanisms

(Relative Velocity Method)

Introduction

- There are various methods of performing kinematic analysis of mechanisms, including *graphical, analytical,* and *numerical*. The choice of a method depends on the problem at hand and on available computational means.
- One way to analyze kinematics of mechanisms is relative velocities and accelerations method.
- Another sufficient method for simple and many compound mechanisms is ***loop-closure equation*** method which is presented here in vector notation.

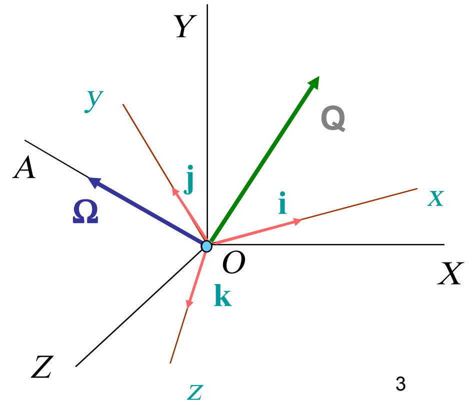
Relative Velocity

- From preliminary dynamics we know that the rate of change of a general vector **Q** with respect to a fixed frame *OXYZ* and with respect to a frame *Oxyz* rotating with an angular velocity **Ω** is

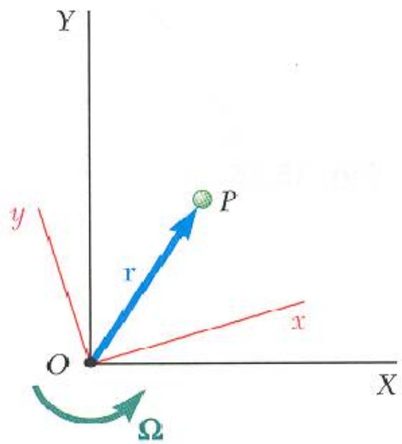
$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \mathbf{\Omega} \times \mathbf{Q}$$

OR

$$\frac{DQ}{Dt} = \frac{dQ}{dt} + \omega \times Q$$



- If frame OXYZ is fixed and frame Oxyz rotates with angular velocity $\vec{\Omega}$, then, Position vector $\vec{r}$ for the particle P is the same in both frames but the rate of change depends on the choice of frame.



- The absolute velocity of the particle P is

$$\vec{v}_P = \left(\dot{\vec{r}}\right)_{OXYZ} = \vec{\Omega} \times \vec{r} + \left(\dot{r}\right)_{Oxyz}$$

- Absolute acceleration for the particle P is

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} + \left(\ddot{\vec{r}}\right)_{Oxyz}$$

The *most general motion of a rigid body in space is equivalent, to the sum of a translation and a rotation.* Considering two particles *A* and *B* of the body:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation. Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, we write

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

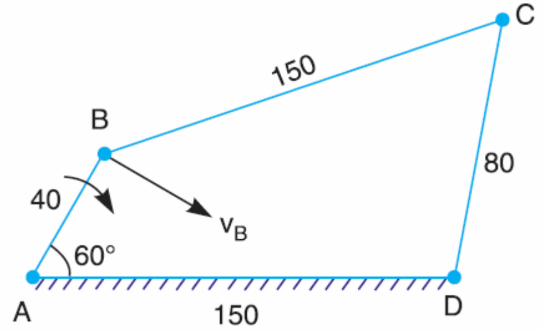
where $\boldsymbol{\omega}$ is the angular velocity of the body at the instant considered. The acceleration of B is, by similar reasoning

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

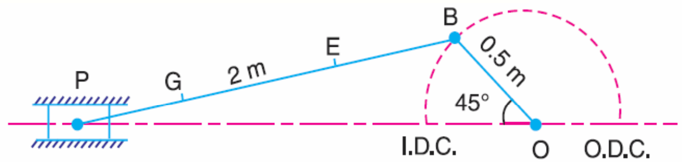
or

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

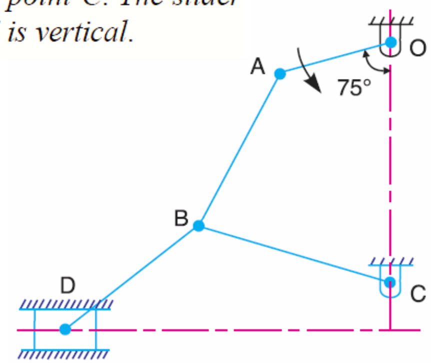
Example : *In a four bar chain ABCD, AD is fixed and is 150 mm long. The crank AB is 40 mm long and rotates at 120 r.p.m. clockwise, while the link CD = 80 mm oscillates about D. BC and AD are of equal length. Find the angular velocity of link CD when angle BAD = 60°.*



Example : *The crank and connecting rod of a theoretical steam engine are 0.5 m and 2 m long respectively. The crank makes 180 r.p.m. in the clockwise direction. When it has turned 45° from the inner dead centre position, determine :* **1.** *velocity of piston,* **2.** *angular velocity of connecting rod,* **3.** *velocity of point E on the connecting rod 1.5 m from the gudgeon pin,* **4.** *velocities of rubbing at the pins of the crank shaft, crank and crosshead when the diameters of their pins are 50 mm, 60 mm and 30 mm respectively,* **5.** *position and linear velocity of any point G on the connecting rod which has the least velocity relative to crank shaft.*

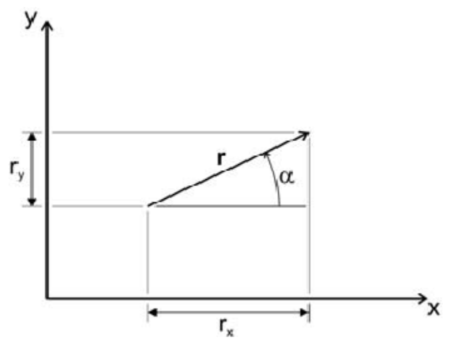


Example : *In Fig. 7.9, the angular velocity of the crank OA is 600 r.p.m. Determine the linear velocity of the slider D and the angular velocity of the link BD, when the crank is inclined at an angle of 75° to the vertical. The dimensions of various links are : OA = 28 mm ; AB = 44 mm ; BC 49 mm ; and BD = 46 mm. The centre distance between the centres of rotation O and C is 65 mm. The path of travel of the slider is 11 mm below the fixed point C. The slider moves along a horizontal path and OC is vertical.*



Vector Algebra and Analysis

A vector **r** has two components in the (x,y) plane: r_x and r_y . Note that the bold font identifies vectors. The following two notations for a vector in a component form will be used:



$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = (r_x, r_y)^T$$

Or

$$\mathbf{r} = |\mathbf{r}|\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = |\mathbf{r}|(\cos\alpha,\ \sin\alpha)^T$$

- An addition/subtraction of two (or more) vectors is a vector whose elements are found by addition/subtraction of the corresponding *x*- and *y*-components of the original vectors. If **a** = $(ax, ay)^T$ and **b** = $(bx, by)^T$, then

$$\mathbf{a} + \mathbf{b} = (a_x + b_x,\ a_y + b_y)^T$$

- The scalar (or dot) product of two vectors is a scalar, which is found by multiplication of the corresponding *x*- and *y*-components of two vectors and then the summation of results. For the two vectors **a** and **b**, their scalar product is

$$\mathbf{d} = \mathbf{a}^T\mathbf{b} = (a_x, a_y)\begin{bmatrix} b_x \\ b_y \end{bmatrix} = a_x b_x + a_y b_y$$

- If vectors **a** and **b** are given in the form **a** = $a\ (\cos\ \alpha,\ \sin\ \alpha)^T$ **b** = , the $b\ (\cos\ \beta,\ \sin\ \beta)^T$ scalar product takes the form

$$\mathbf{d} = \mathbf{a}^T\mathbf{b} = ab(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = ab\cos(\alpha - \beta)$$

The result of the cross-product of two vectors is a vector perpendicular to the plane in which the original two vectors are lying. Thus, if the two vectors are lying in the (x,y) plane, then their product will have a *z*-direction.

$$\mathbf{u} = \mathbf{a}^T \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{bmatrix}$$

$$\mathbf{u} = (a_y 0 - b_y 0)\ \mathbf{i} - (a_x 0 - b_x 0)\ \mathbf{j} + (a_x b_y - b_x a_y)\ \mathbf{k} = (a_x b_y - b_x a_y)\ \mathbf{k}$$

$$|\mathbf{u}| = u = (a_x b_y - b_x a_y)$$

$$\mathbf{u} = ab\ (\cos\ \alpha\ \sin\ \beta - \cos\ \beta\ \sin\ \alpha) = ab\ \sin(\beta - \alpha)$$

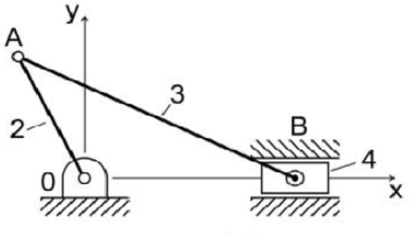
$$\mathbf{a} = a(t)\ [\cos\ \alpha(t),\ \sin\ \alpha(t)]^T,$$

$$\frac{d\mathbf{a}(t)}{dt} = \frac{da(t)}{dt}[\cos\alpha(t),\ \sin\alpha(t)]^T + a(\mathbf{t})[-\sin\alpha(t),\ \cos\alpha(t)]^T\frac{d\alpha(t)}{dt}$$

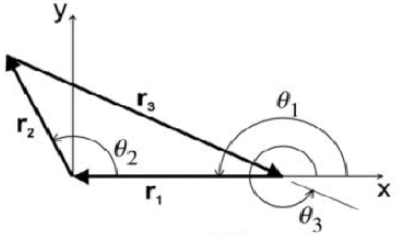
Kinematic Analysis Of The Slider-crank Mechanism Using Loop-closure Method

$$\sum_{i=1}^{3} \mathbf{r}_i = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0$$

$$\sum_{i=1}^{N} r_i(\cos\theta_i, \sin\theta_i)^T = 0$$



(a)



(b)

- In order to perform a velocity analysis, we can derivate position equation with respect to time:

$$\sum_{i=1}^{N} \dot{r}_i(t)[\cos\theta_i(t), \sin\theta_i(t)]^T + r_i(t)[-\sin\theta_i(t), \cos\theta_i(t)]^T\omega_i(t) = 0$$

- And for acceleration analysis:

$$\sum_{i=1}^{N} \ddot{r}_i[\cos\theta_i, \sin\theta_i]^T + 2\dot{r}_i[-\sin\theta_i, \cos\theta_i]^T\omega_i$$

$$-r_i[\cos\theta_i, \sin\theta_i]^T\omega_i^2 + r_i[-\sin\theta_i, \cos\theta_i]^T\alpha_i = 0$$

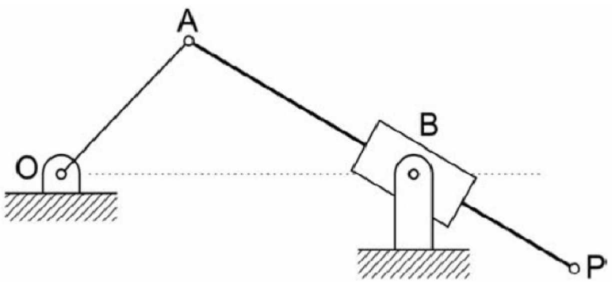
Example 1: In below figure an inverted slider-crank mechanism is shown.

a. Write a loop-closure equation for this mechanism.

b. If the input is the crank angle, what are the unknowns?

c. Solve the equation for the unknowns.

d. Express the position of point *P* in terms of the input angle.



- **Example 2:** In below figure a slider-crank mechanism is shown.
- a. Write a loop-closure equation for this mechanism.
- b. If the input is the crank angle, what are the unknowns?
- c. Solve the equation for the unknowns.
- d. Express the position of point *P* in terms of the input angle.