Chapter 9:
Moments of Inertia

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Introduction

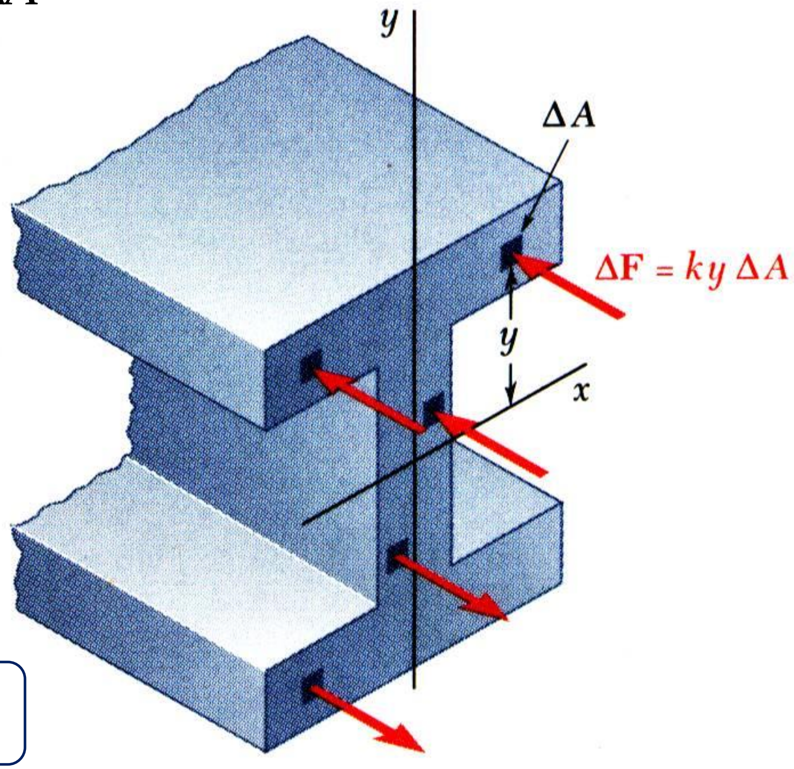
- Previously considered distributed forces which were proportional to the area or volume over which they act.
  - The resultant was obtained by summing or integrating over the areas or volumes.
  - The moment of the resultant about any axis was determined by computing the first moments of the areas or volumes about that axis.

- Will now consider forces which are proportional to the area or volume over which they act but also vary linearly with distance from a given axis.
  - It will be shown that the magnitude of the resultant depends on the first moment of the force distribution with respect to the axis.
  - The point of application of the resultant depends on the second moment of the distribution with respect to the axis.

- Current chapter will present methods for computing the moments and products of inertia for areas and masses.

Moment of Inertia of an Area

- Consider distributed forces $\Delta\vec{F}$ whose magnitudes are proportional to the elemental areas ΔA on which they act and also vary linearly with the distance of ΔA from a given axis.



- Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta\vec{F} = ky\Delta A \quad \Rightarrow$$

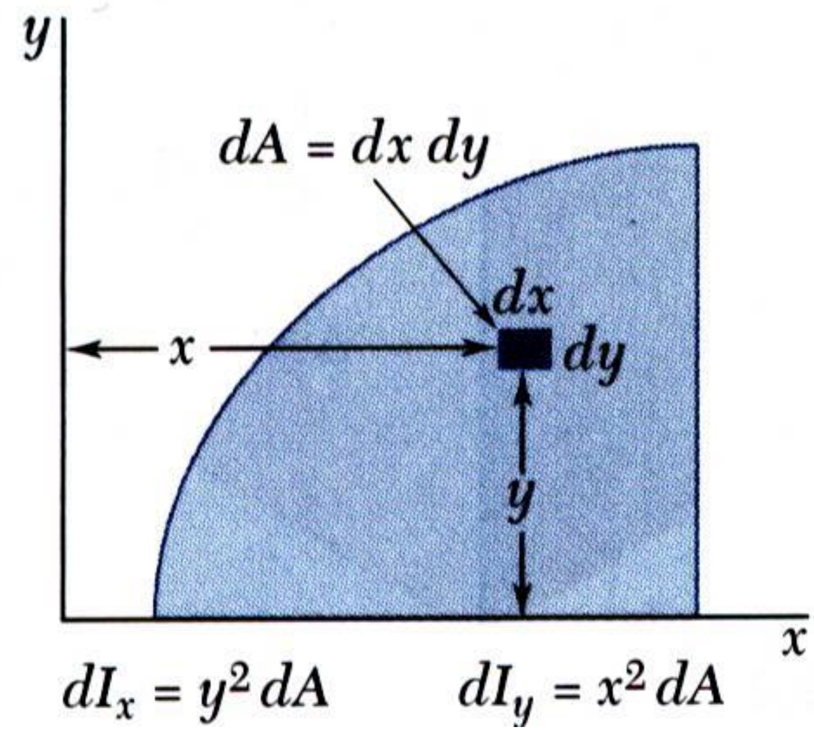
$$R = \int \Delta F = k\int y\,dA = 0 \qquad \int y\,dA = Q_x = \text{first moment}$$

$$M = \int \Delta F \cdot y = k\int y^2 dA \qquad \int y^2 dA = \text{second moment}$$

Moment of Inertia of an Area by Integration

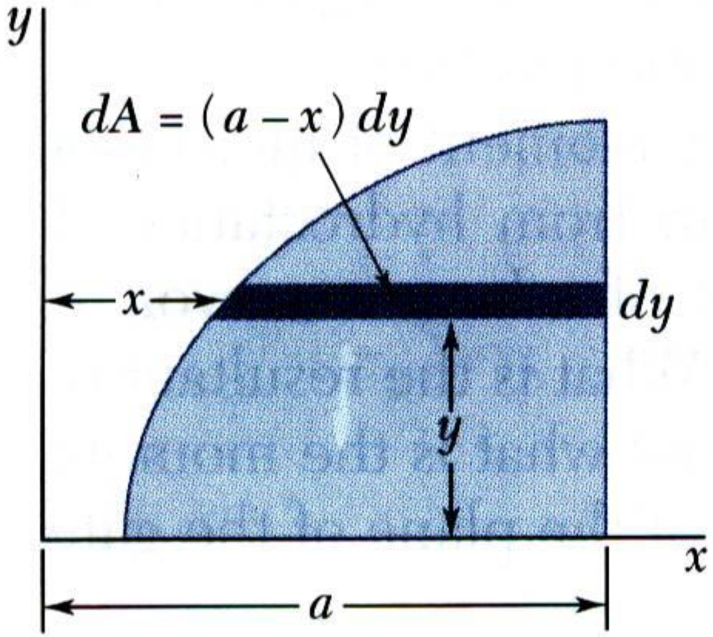
- *Second moments* or *moments of inertia* of an area with respect to the *x* and *y* axes,

$$I_x = \int y^2 dA \quad , \quad I_y = \int x^2 dA$$

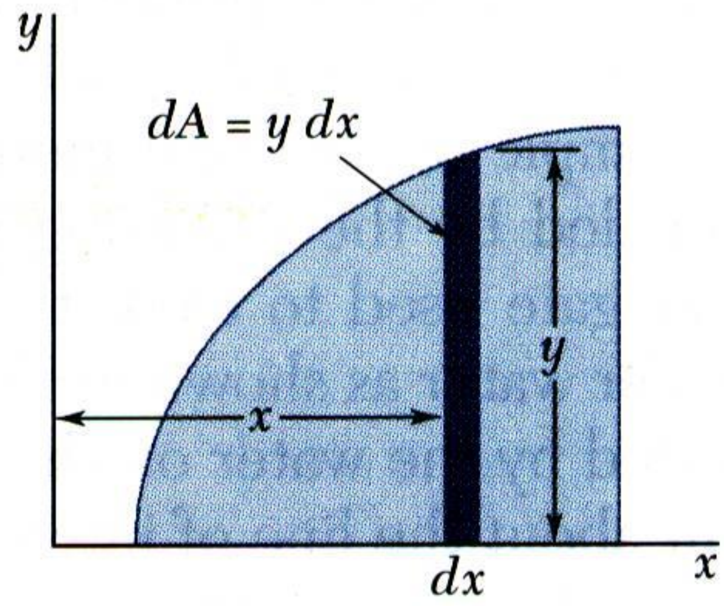


Moment of Inertia of an Area by Integration

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.



$dI_x = y^2\,dA$



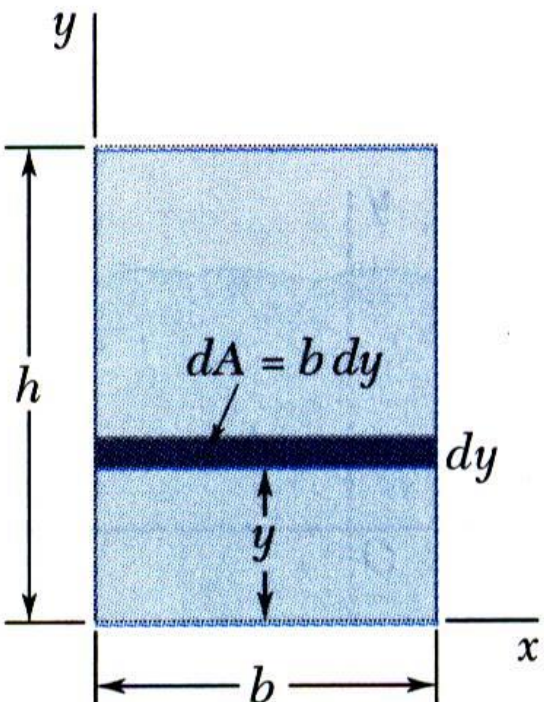
$dI_y = x^2\,dA$

Moment of Inertia of an Area by Integration

- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \left.\frac{by^3}{3}\right]_0^h$$

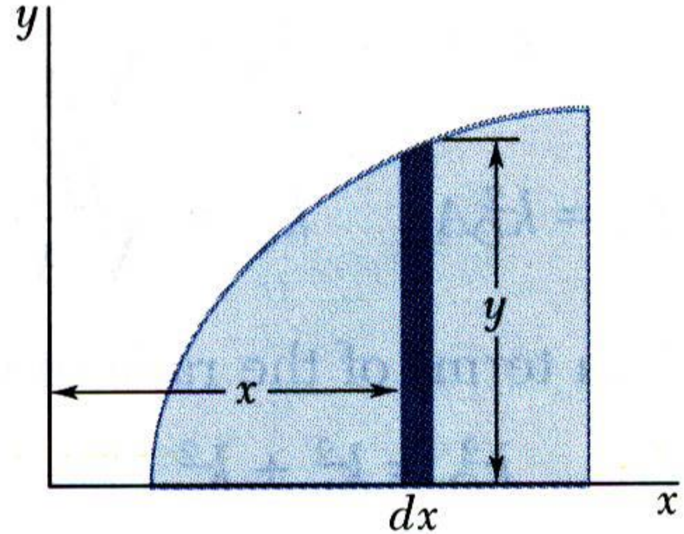
$$\Rightarrow \boxed{I_x = \frac{1}{3}bh^3}$$



❑ Moment of Inertia of an Area by Integration

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \frac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y \, dx$$

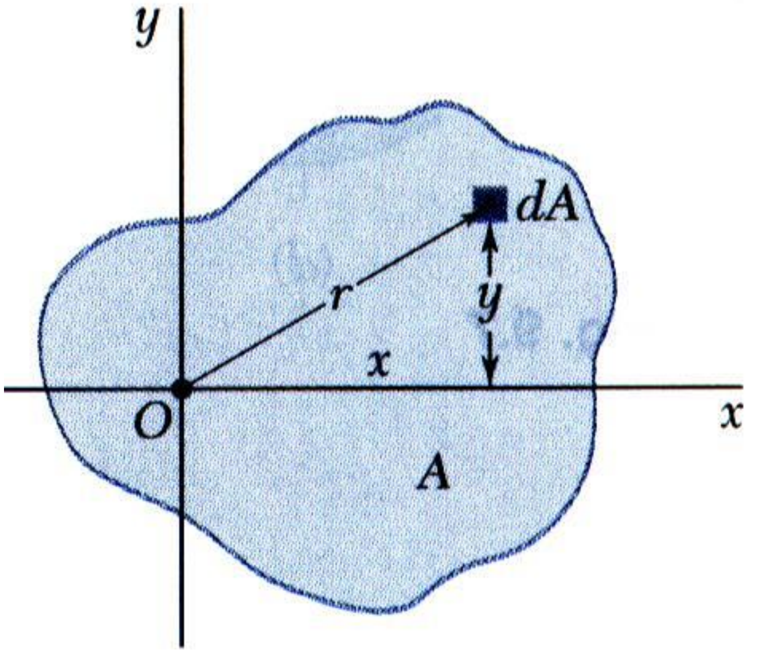


$$dI_x = \frac{1}{3} y^3 \, dx$$

$$dI_y = x^2 y \, dx$$

Polar Moment of Inertia

- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.



$$\boxed{J_0 = \int r^2 dA}$$

- The polar moment of inertia is related to the rectangular moments of inertia,

$$J_0 = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA$$

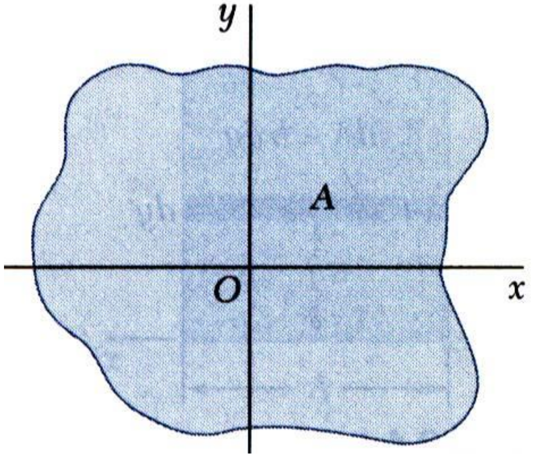
$$\Rightarrow \boxed{J_0 = I_y + I_x}$$

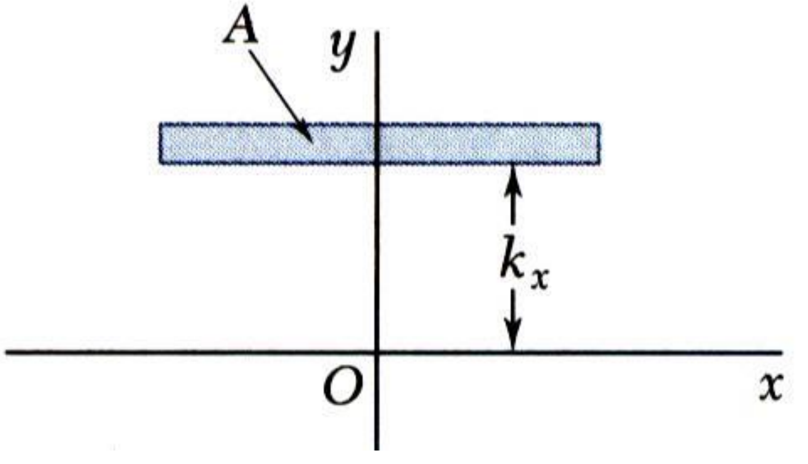
Radius of Gyration of an Area

- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \quad \Rightarrow \quad k_x = \sqrt{\frac{I_x}{A}}$$

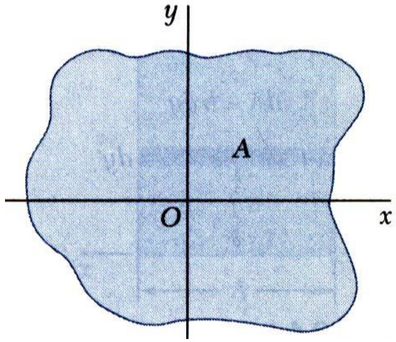
$k_x =$ *radius of gyration* with respect to the x axis



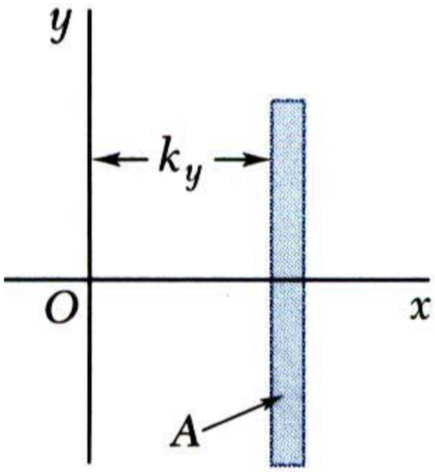


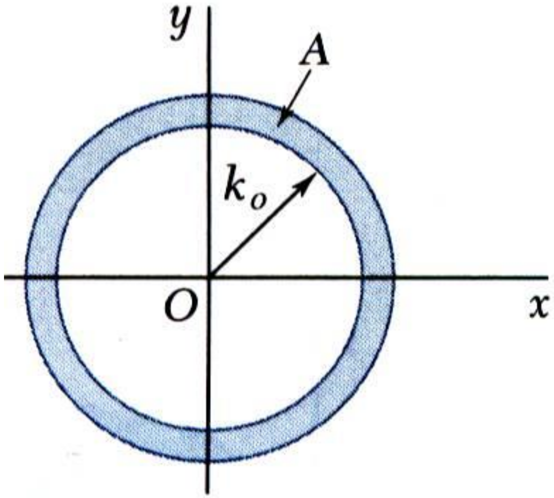
Radius of Gyration of an Area

- Similarly,



$$\boxed{I_y = k_y^2 A \quad \Rightarrow \quad k_y = \sqrt{\frac{I_y}{A}}}$$



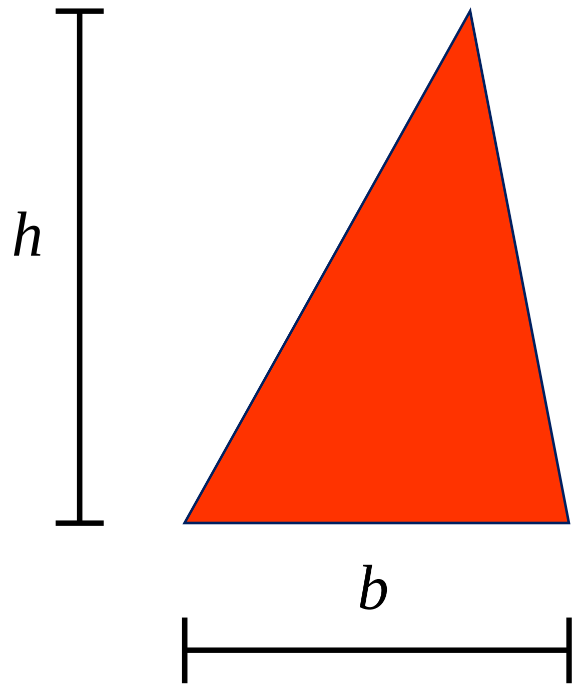


$$\boxed{J_O = k_O^2 A \quad \Rightarrow \quad k_O = \sqrt{\frac{J_O}{A}}}$$

$$\boxed{k_O^2 = k_x^2 + k_y^2}$$

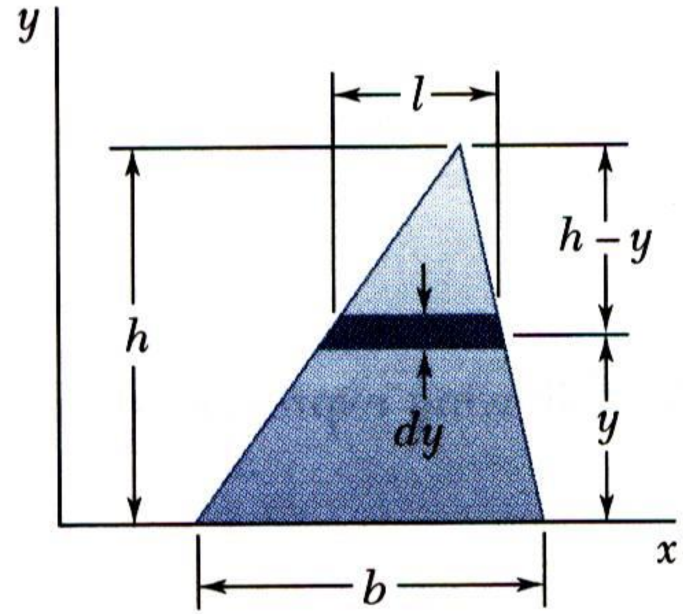
❑ Sample Problem 01

Determine the moment of inertia of a triangle with respect to its base.



❑ Sample Problem 01

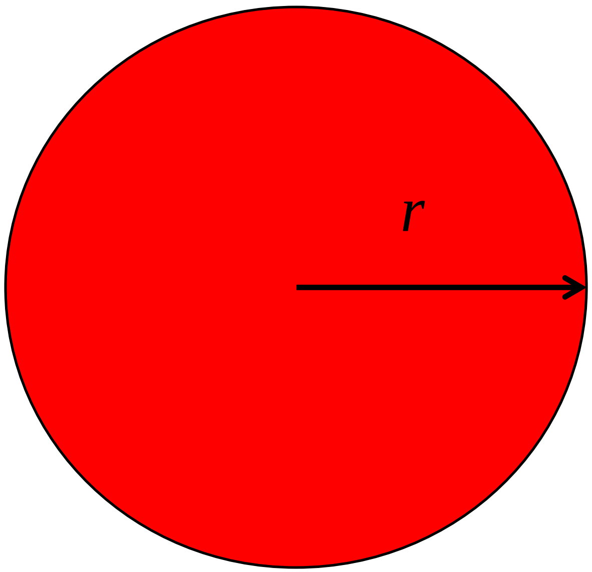
SOLUTION:



❑ Sample Problem 02

a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

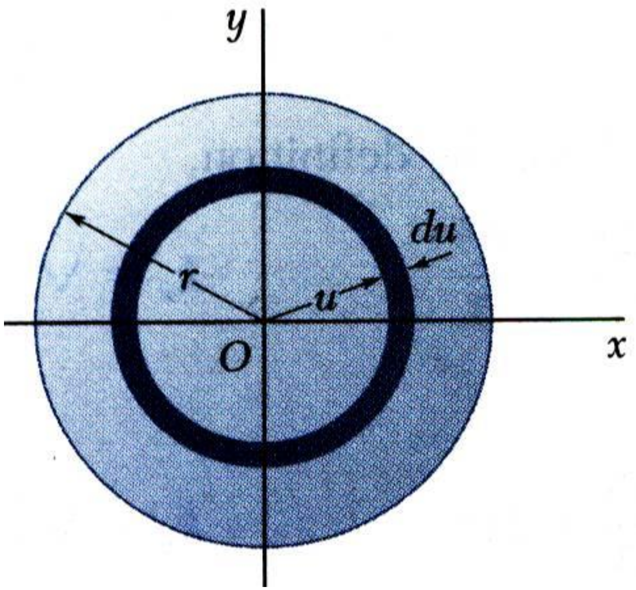
b) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.



❑ Sample Problem 02

SOLUTION:

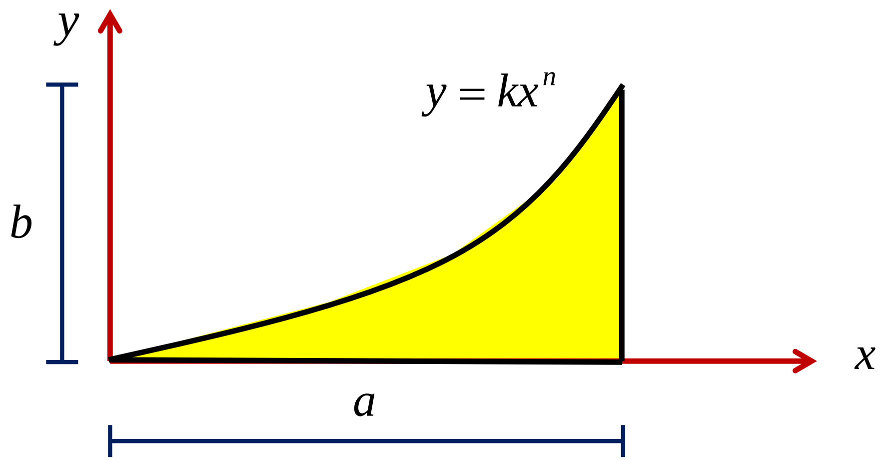
- An annular differential area element is chosen,



❑ Sample Problem 03

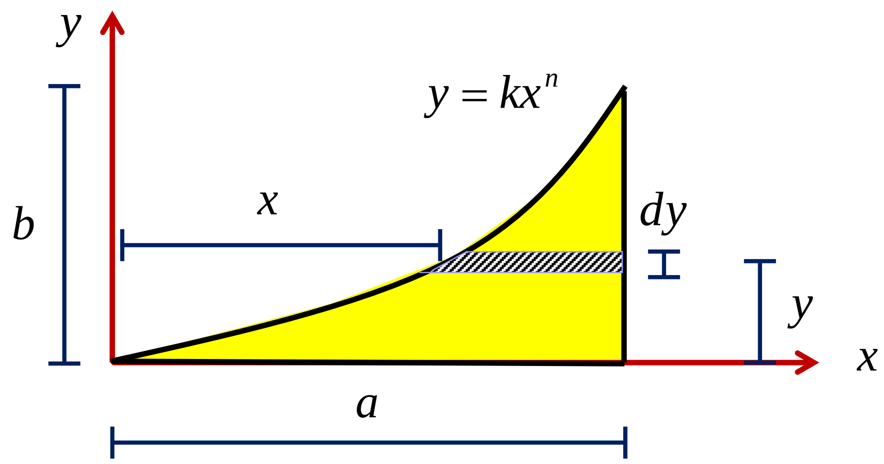
(a) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axe.

(b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.



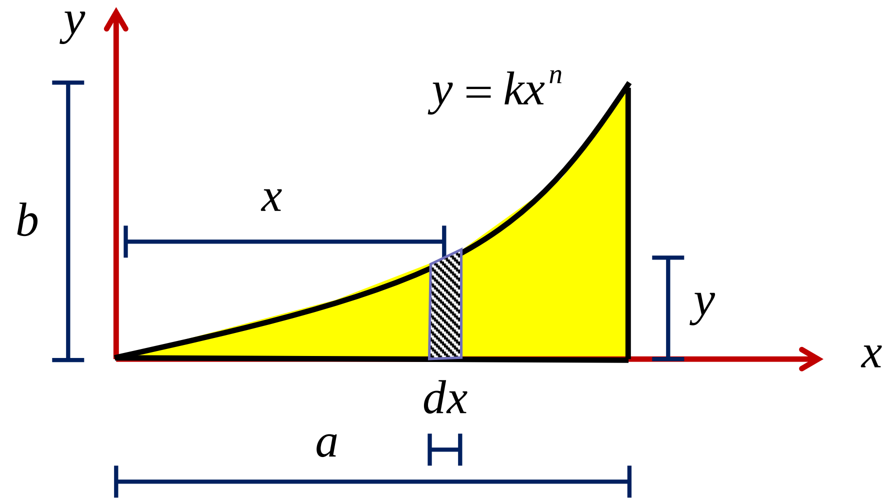
Sample Problem 03

SOLUTION:



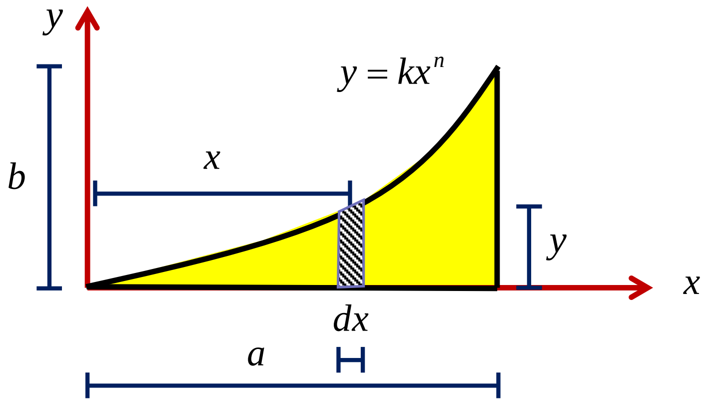
❑ Sample Problem 03

SOLUTION:



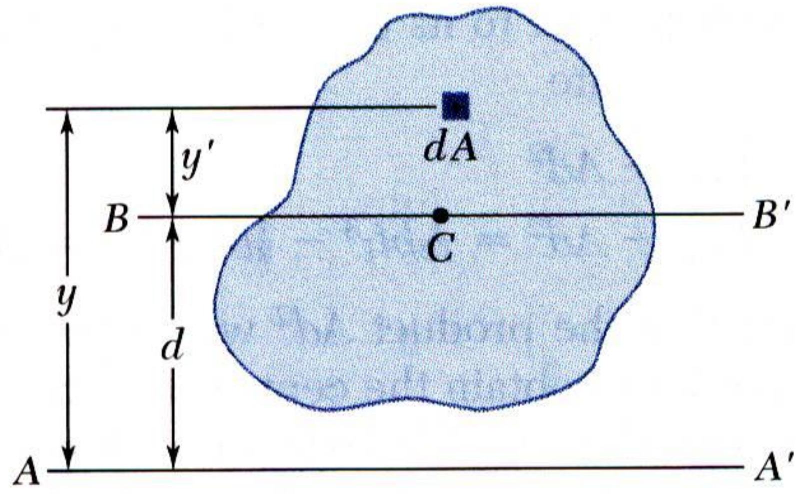
❑ Sample Problem 03

SOLUTION:



Parallel Axis Theorem

- Consider moment of inertia I of an area A with respect to the axis AA'



$$I = \int y^2 dA$$

- The axis BB' passes through the area centroid and is called a ***centroidal axis***.

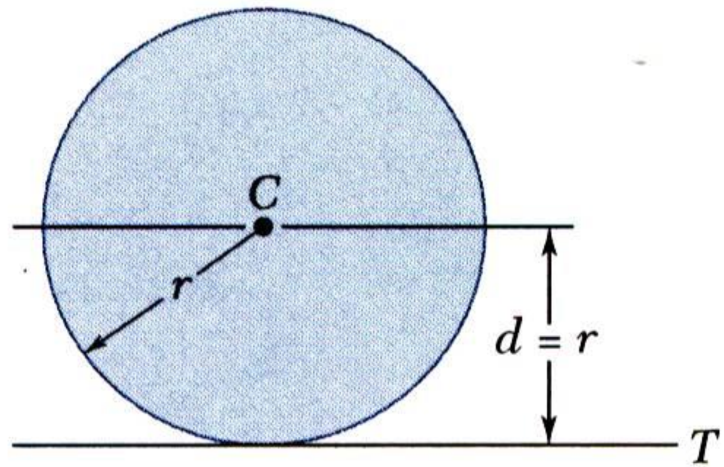
$$I = \int y^2 dA = \int (y' + d)^2 dA = \int y'^2 dA + 2d \int y' dA + d^2 \int dA \quad \Rightarrow \quad \boxed{I = \bar{I} + Ad^2}$$

parallel axis theorem

❑ Parallel Axis Theorem

- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \frac{1}{4}\pi r^4 + (\pi r^2)\, r^2 \quad \Rightarrow \quad \boxed{I_T = \frac{5}{4}\pi r^4}$$

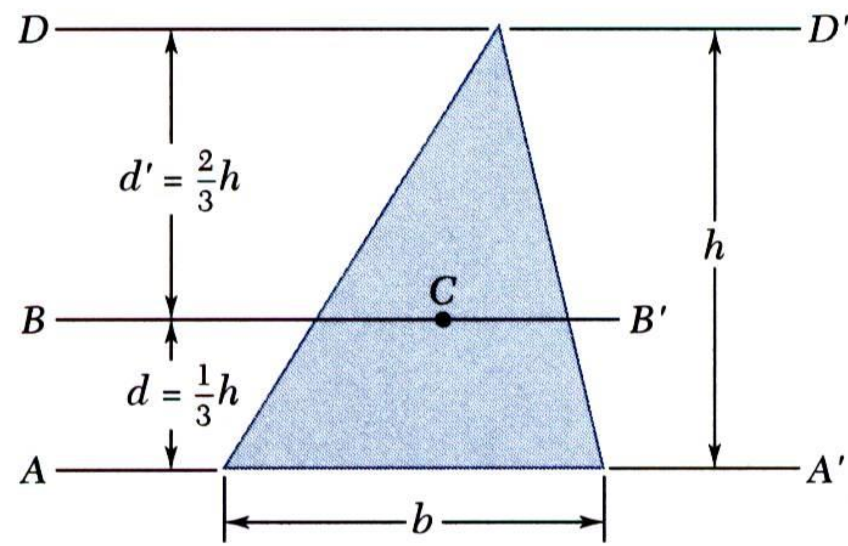


- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2 \quad \Rightarrow$$

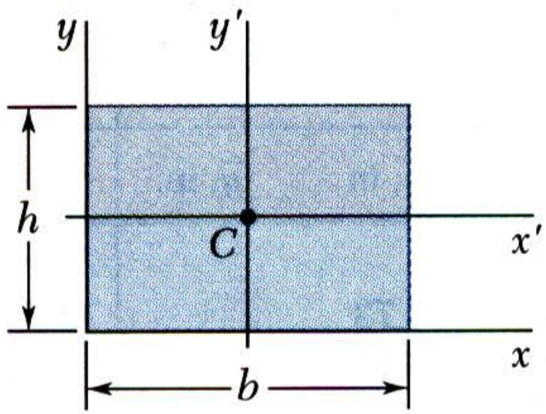
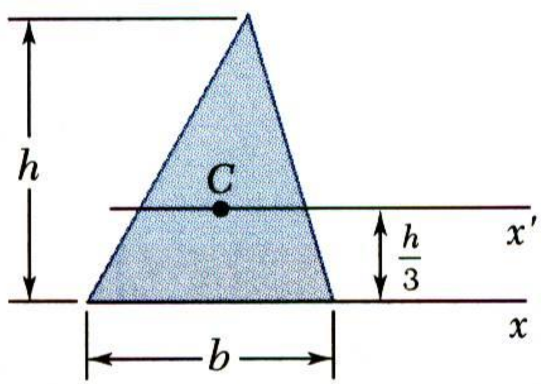
$$I_{BB'} = I_{AA'} - Ad^2 = \frac{1}{12}bh^3 - \frac{1}{2}bh\left(\frac{1}{3}h\right)^2$$

$$\Rightarrow \quad \boxed{I_{BB'} = \frac{1}{36}bh^3}$$

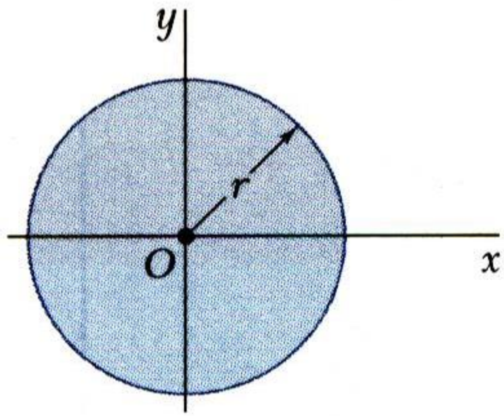
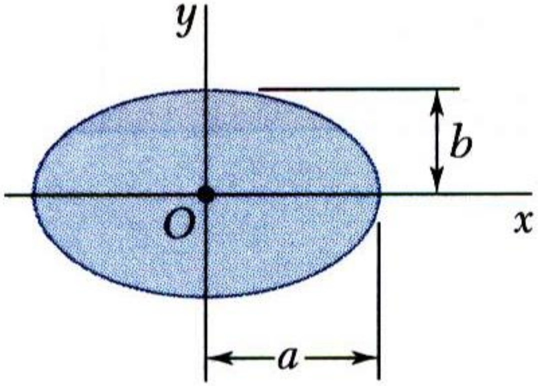


Moments of Inertia of Composite Areas

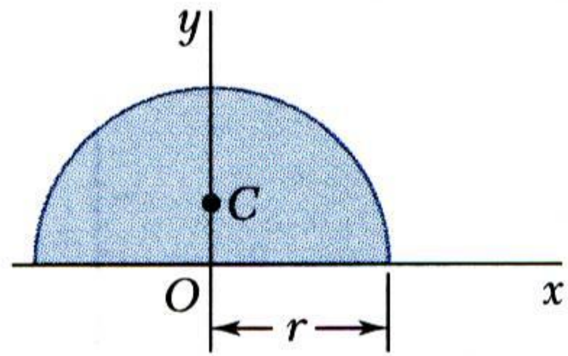
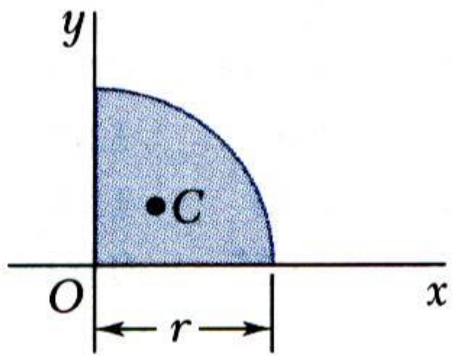
- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$

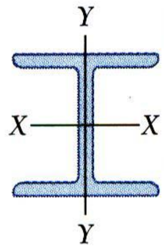
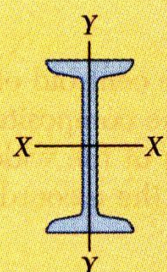
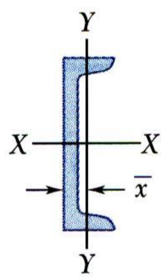
Moments of Inertia of Composite Areas

Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi a b^3$ $\bar{I}_y = \frac{1}{4}\pi a^3 b$ $J_O = \frac{1}{4}\pi a b(a^2 + b^2)$

❑ Moments of Inertia of Composite Areas

Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$

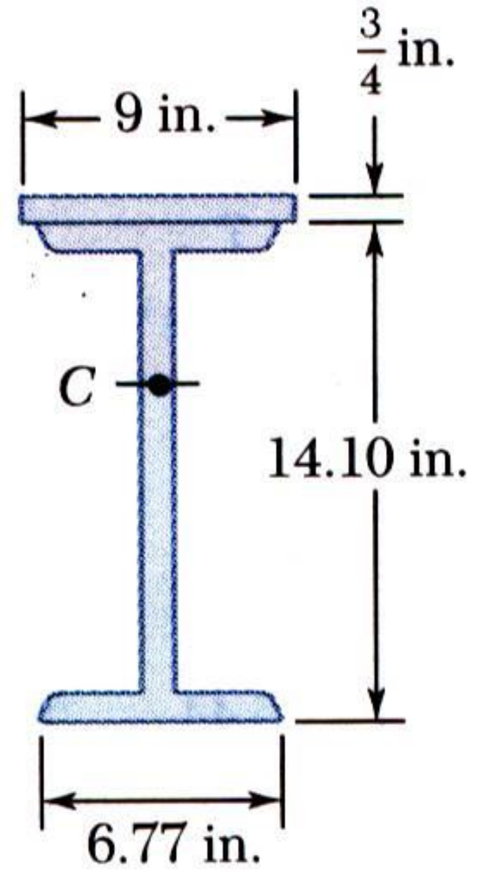
Moments of Inertia of Composite Areas

Shape	Designation	Area mm²	Depth mm	Width mm	Axis X-X $\bar{I}_x$ 10^6 mm⁴	Axis X-X $\bar{k}_x$ mm	Axis X-X $\bar{y}$ mm	Axis Y-Y $\bar{I}_y$ 10^6 mm⁴	Axis Y-Y $\bar{k}_y$ mm	Axis Y-Y $\bar{x}$ mm
W Shapes (Wide-Flange Shapes) 	W460 × 113†	14400	463	280	554	196.3		63.3	66.3	
	W410 × 85	10800	417	181	316	170.7		17.94	40.6	
	W360 × 57	7230	358	172	160.2	149.4		11.11	39.4	
	W200 × 46.1	5890	203	203	45.8	88.1		15.44	51.3	
S Shapes (American Standard Shapes) 	S460 × 81.4†	10390	457	152	335	179.6		8.66	29.0	
	S310 × 47.3	6032	305	127	90.7	122.7		3.90	25.4	
	S250 × 37.8	4806	254	118	51.6	103.4		2.83	24.2	
	S150 × 18.6	2362	152	84	9.2	62.2		0.758	17.91	
C Shapes (American Standard Channels) 	C310 × 30.8†	3929	305	74	53.7	117.1		1.615	20.29	17.73
	C250 × 22.8	2897	254	65	28.1	98.3		0.949	18.11	16.10
	C200 × 17.1	2181	203	57	13.57	79.0		0.549	15.88	14.50
	C150 × 12.2	1548	152	48	5.45	59.4		0.288	13.64	13.00

❑ Sample Problem 04

The strength of a W14x38 rolled steel beam is increased by attaching a plate to its upper flange.

Determine the moment of inertia and radius of gyration with respect to an axis which is parallel to the plate and passes through the centroid of the section.



W14×38:

$A = 11.20\ (\text{in}^2)$

$\bar{I}_x = 385\ (\text{in}^4)$

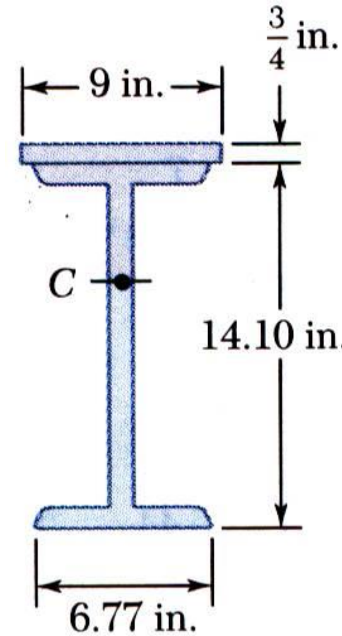
❑ Sample Problem 04

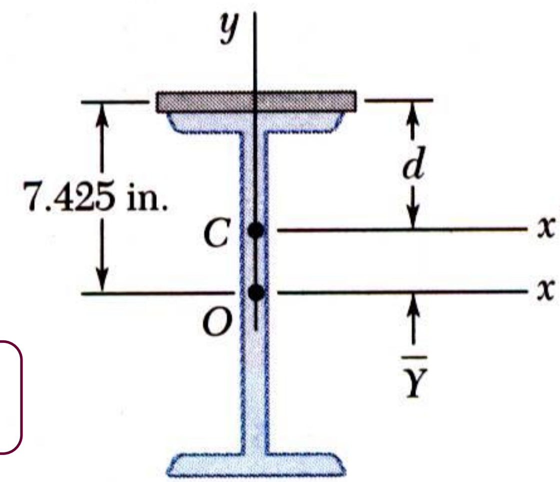
SOLUTION:

- Determine location of **the centroid of composite section** with respect to a coordinate system with **origin at the centroid of the beam section.**

Section	$A\,(\text{in}^2)$	$\bar{y}$ (in.)	$\bar{y}A\,(\text{in}^3)$
Plate	$9\times\frac{3}{4}=6.75$	$\frac{14.10}{2}+\frac{1}{2}\times\frac{3}{4}=7.425$	50.12
Beam	11.20	0	0
	$\boxed{\sum A=17.95}$		$\boxed{\sum \bar{y}A=50.12}$

$$\bar{Y}\sum A=\sum \bar{y}A \quad\Rightarrow\quad \bar{Y}=\frac{\sum \bar{y}A}{\sum A}=\frac{50.12\,(\text{in}^3)}{17.95\,(\text{in}^2)} \quad\Rightarrow\quad \boxed{\bar{Y}=2.792\,(\text{in.})}$$





Sample Problem 04

SOLUTION:

- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

$$I_{x',\text{beam}} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.20)(2.792)^2$$

$$\Rightarrow \boxed{I_{x',\text{beam}} = 472.3\ (\text{in}^4)}$$

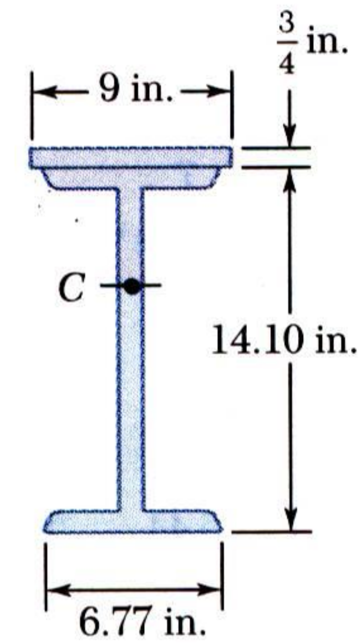
$$I_{x',\text{plate}} = \bar{I}_x + Ad^2 = \frac{1}{12}(9)\left(\frac{3}{4}\right)^3 + (6.75)(7.425 - 2.792)^2$$

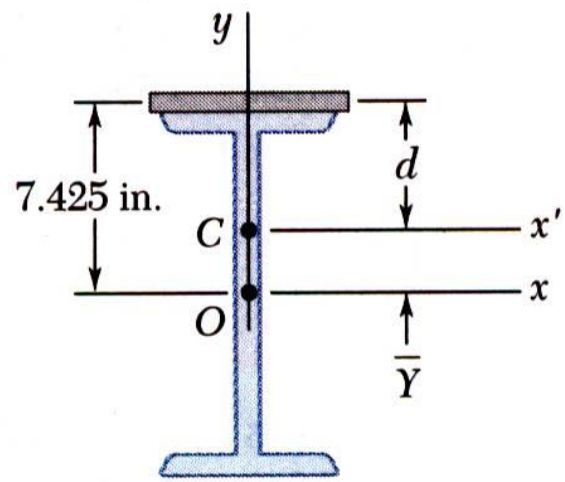
$$\Rightarrow \boxed{I_{x',\text{plate}} = 145.2\ (\text{in}^4)}$$

$$I_{x'} = I_{x',\text{beam}} + I_{x',\text{plate}} = 472.3 + 145.2 \Rightarrow \boxed{I_{x'} = 617.5\ (\text{in}^4)}$$

- Calculate the radius of gyration from the moment of inertia of the composite section.

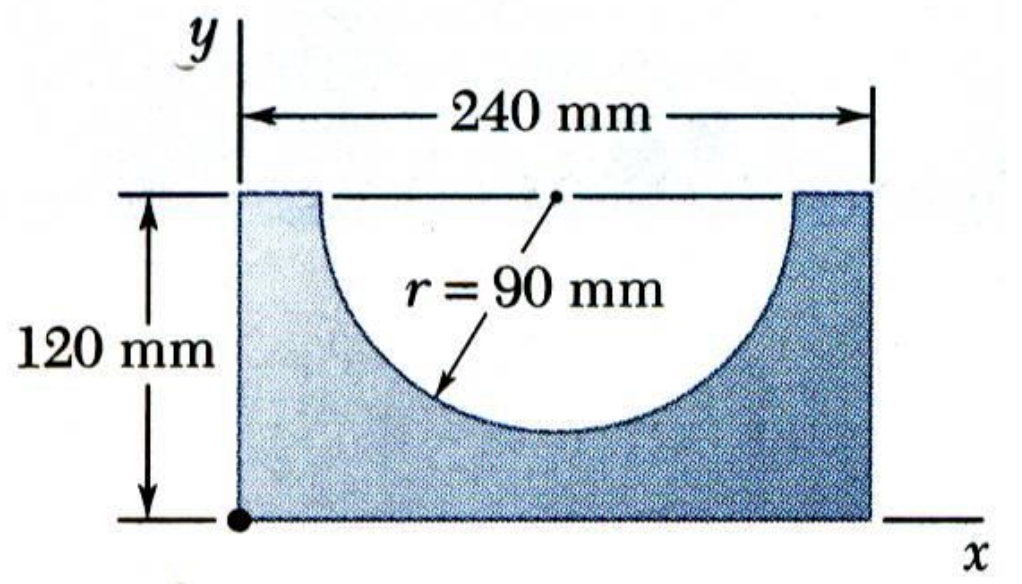
$$k_{x'} = \sqrt{\frac{I_{x'}}{A}} = \sqrt{\frac{617.5\ (\text{in}^4)}{17.95\ (\text{in}^2)}} \Rightarrow \boxed{k_{x'} = 5.87\ (\text{in.})}$$





❑ Sample Problem 05

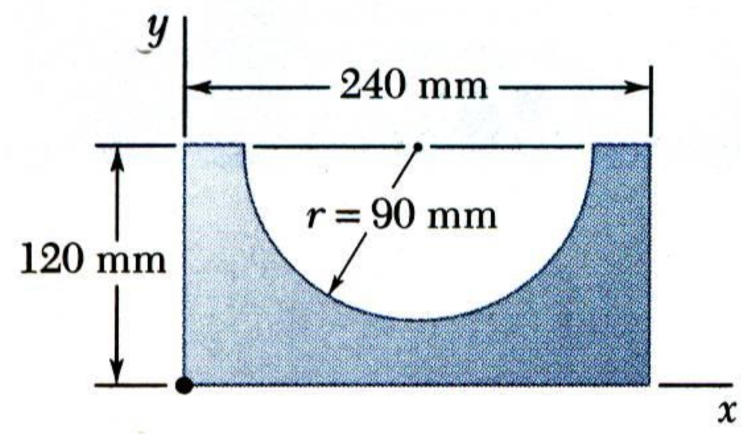
Determine the moment of inertia of the shaded area with respect to the *x* axis.



❑ Sample Problem 05

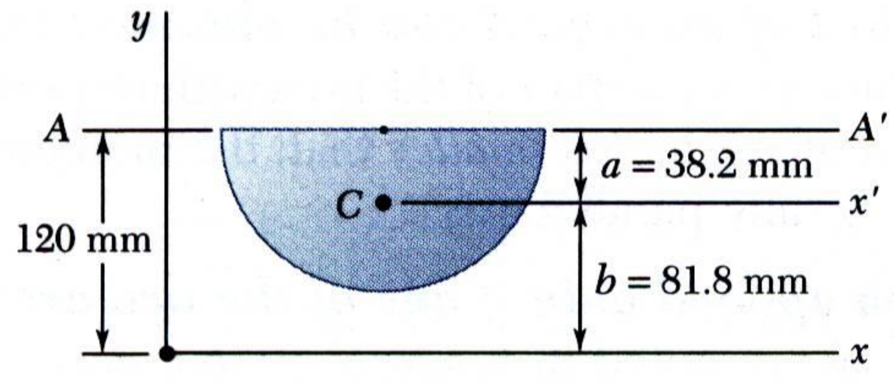
SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.



❑ Sample Problem 05

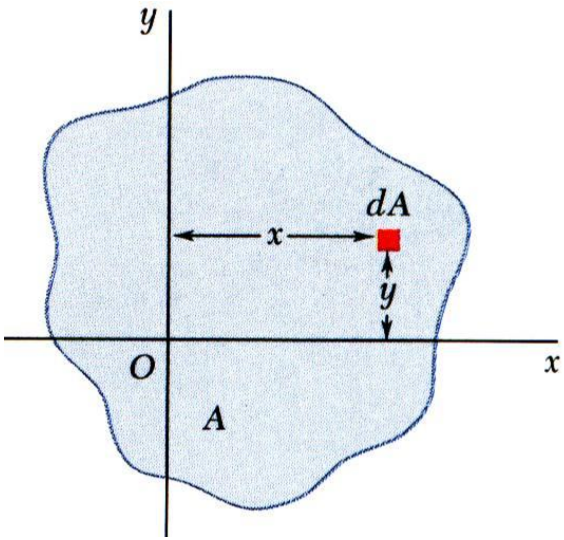
SOLUTION:



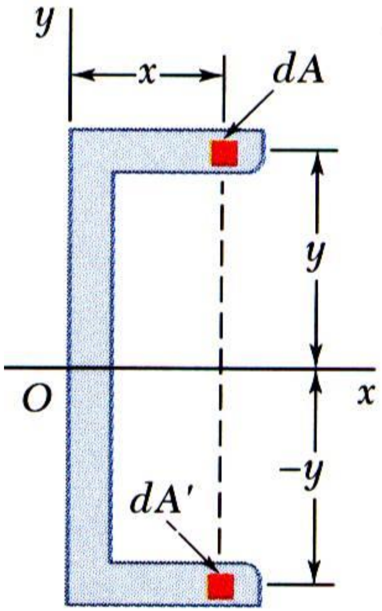
❑ Product of Inertia

- *Product of Inertia:*

$$I_{xy} = \int xy \, dA$$



Unlike the moments of inertia Ix and Iy the product of inertia Ixy can be positive, negative, or zero.



- **When the *x* axis, the *y* axis, or both are an axis of symmetry, the product of inertia is zero**.

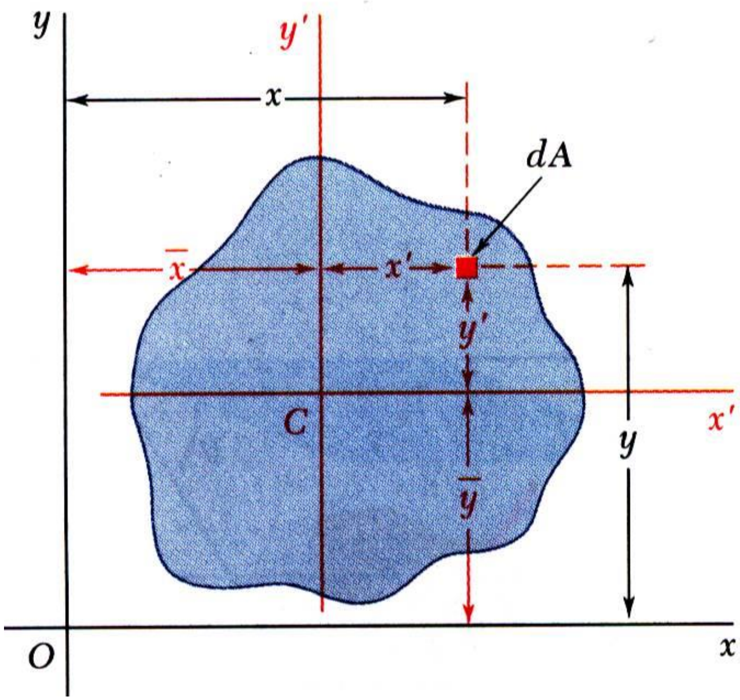
❑ Product of Inertia

- Parallel axis theorem for products of inertia:

$$I_{xy} = \int xy\, dA = \int (x' + \bar{x})(y' + \bar{y})\, dA$$

$$= \int x'y'\, dA + \bar{y}\int \cancel{x'}\, dA + \bar{x}\int \cancel{y'}\, dA + \bar{x}\bar{y}\int dA$$

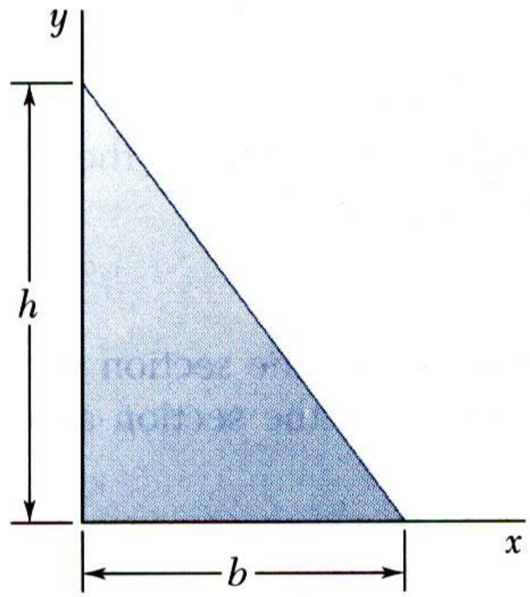
$$\Rightarrow \quad \boxed{I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A}$$



❑ Sample Problem 06

Determine the product of inertia of the right triangle

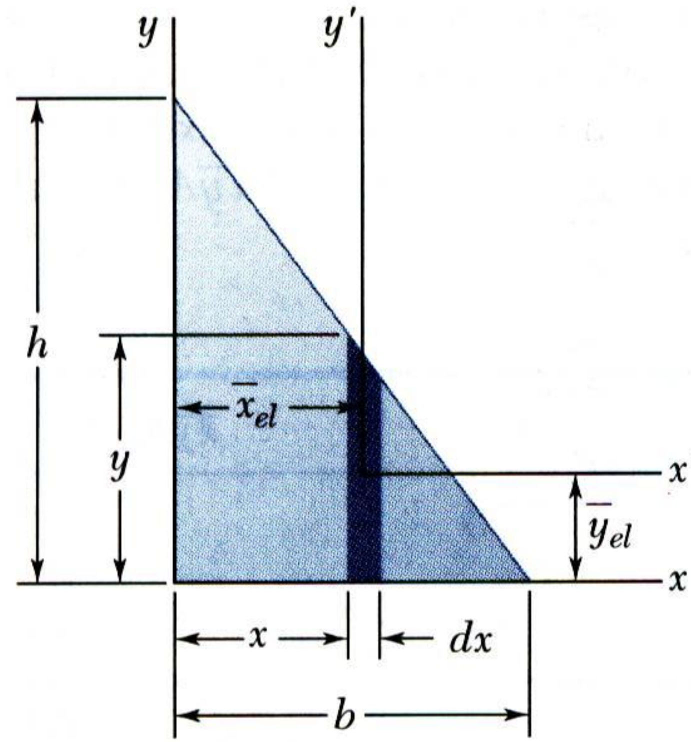
(a) with respect to the *x* and *y* axes and
(b) with respect to centroidal axes parallel to the *x* and *y* axes.



❑ Sample Problem 06

SOLUTION:

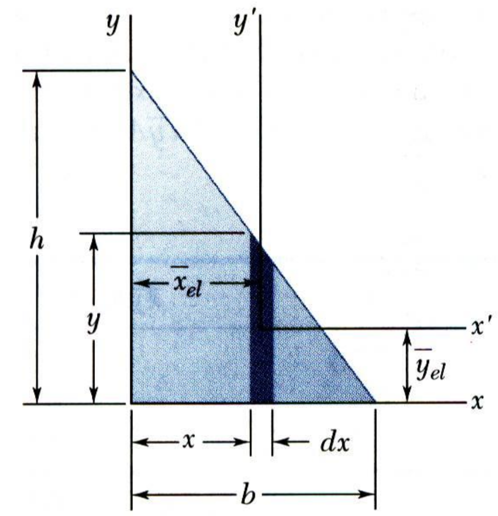
- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips



Sample Problem 06

SOLUTION:

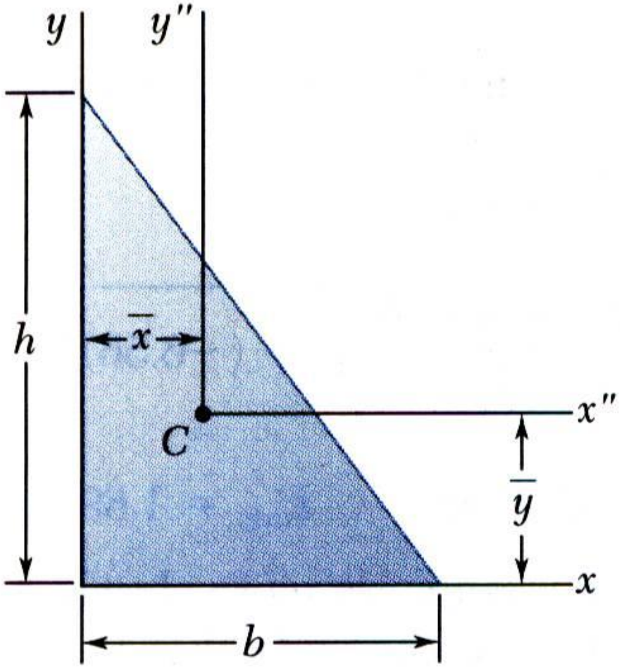
Integrating dI_x from $x = 0$ to $x = b$,



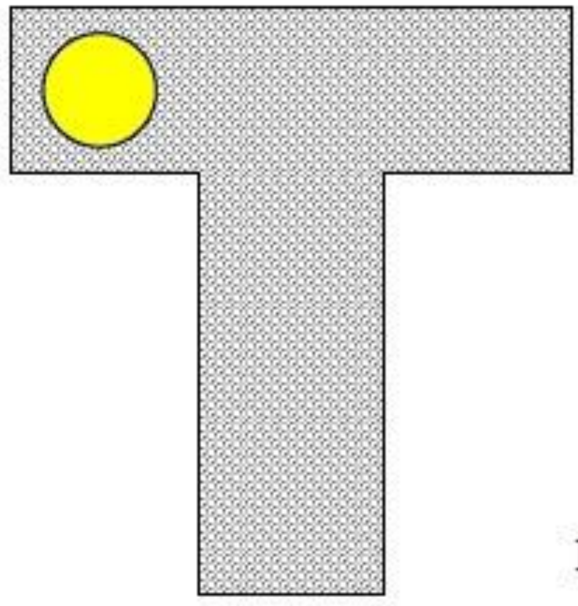
Sample Problem 06

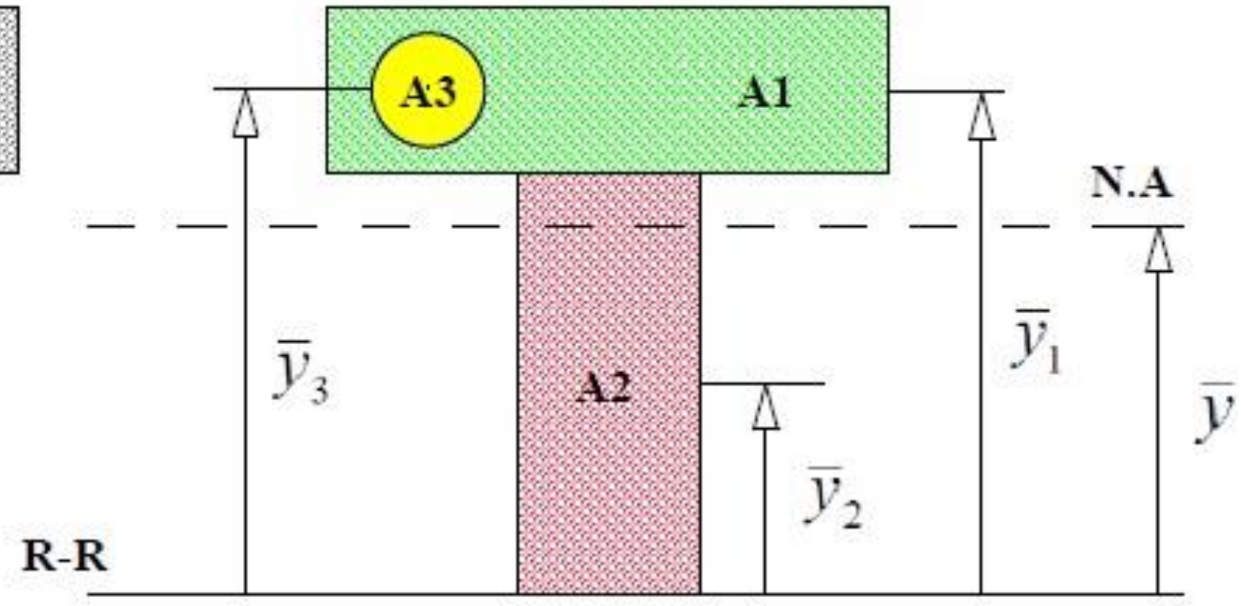
SOLUTION:

- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

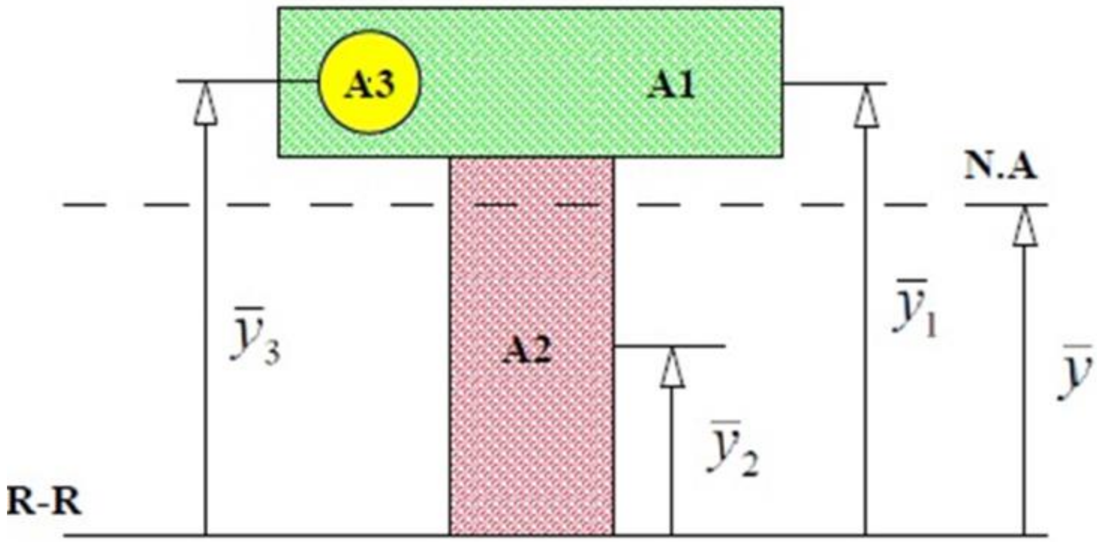


❑ Systematic Calculation of the Moment of Inertia





Systematic Calculation of the Moment of Inertia



Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^{\,2}$	$I_{g\,i}$
1	A_1	$\bar{y}_1$	$A_1\bar{y}_1$	$A_1\bar{y}_1^{\,2}$	$I_{g\,1}$
2	A_2	$\bar{y}_2$	$A_2\bar{y}_2$	$A_2\bar{y}_2^{\,2}$	$I_{g\,2}$
3	$-A_3$	$\bar{y}_3$	$-A_3\bar{y}_3$	$-A_3\bar{y}_3^{\,2}$	$-I_{g\,3}$
	$\sum A_i$		$\sum A_i\bar{y}_i$	$\sum A_i\bar{y}_i^{\,2}$	$\sum I_{g\,i}$

Systematic Calculation of the Moment of Inertia

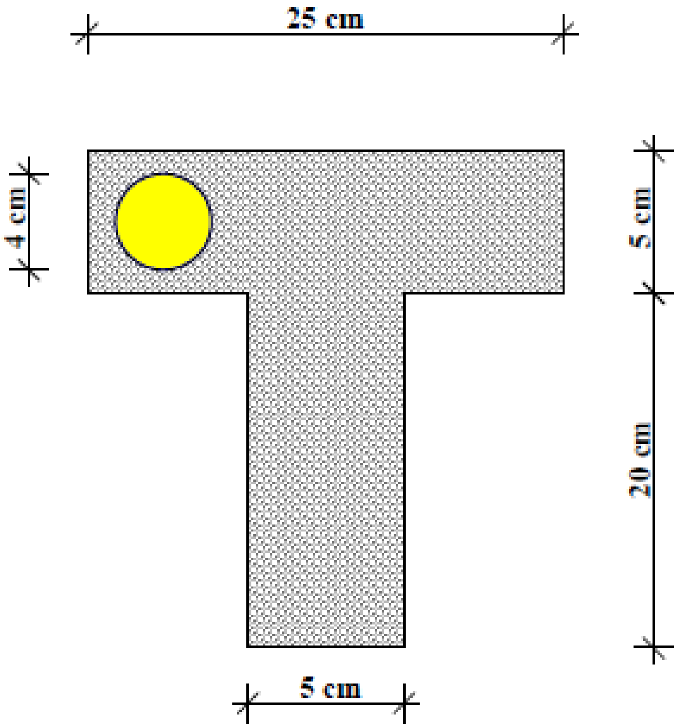
Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^{\,2}$	$I_{g\,i}$
1	A_1	$\bar{y}_1$	$A_1\bar{y}_1$	$A_1\bar{y}_1^{\,2}$	$I_{g\,1}$
2	A_2	$\bar{y}_2$	$A_2\bar{y}_2$	$A_2\bar{y}_2^{\,2}$	$I_{g\,2}$
3	$-A_3$	$\bar{y}_3$	$-A_3\bar{y}_3$	$-A_3\bar{y}_3^{\,2}$	$-I_{g\,3}$
	$\sum A_i$		$\sum A_i\bar{y}_i$	$\sum A_i\bar{y}_i^{\,2}$	$\sum I_{g\,i}$

$$A = \sum A_i \qquad I_{R-R} = \sum I_{g\,i} + \sum A_i\bar{y}_i^{\,2}$$

$$\bar{y} = \frac{\sum A_i\bar{y}_i}{\sum A_i} \qquad I_{NA} = \sum I_{g\,i} + \sum A_i\bar{y}_i^{\,2} - \frac{(\sum A_i\bar{y}_i)^2}{\sum A_i}$$

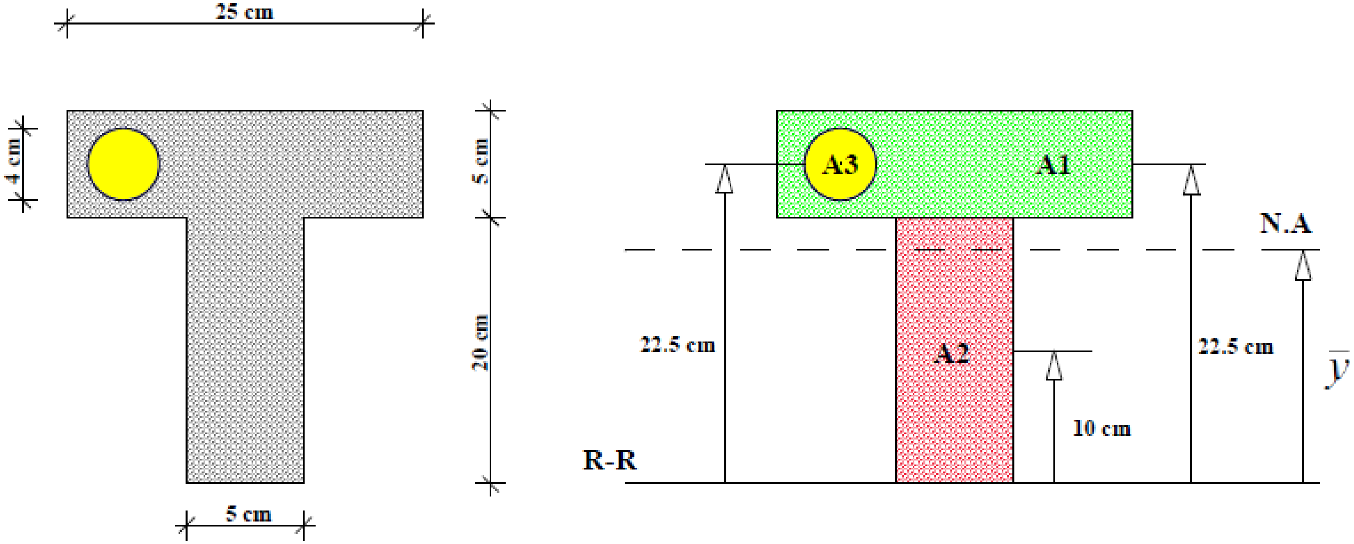
❑ Sample Problem 07

Determine the moment of Inertia.



❑ Sample Problem 07

SOLUTION:



Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^{\,2}$	$I_{g\,i}$
1	$5\times 25=125$	22.5	2812.5	63281.25	$\frac{1}{12}(25)(5)^3=260.42$
2	$5\times 20=100$	10	1000	10000	$\frac{1}{12}(5)(20)^3=3333.33$
3	$-\pi\left(\frac{4}{2}\right)^2=-12.57$	22.5	-282.83	-6363.68	$-\pi\left(\frac{4}{2}\right)^4=-50.27$
	212.43		3529.67	66917.57	3543.48

Sample Problem 07

SOLUTION:

Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^{\,2}$	$I_{g\,i}$
1	$5\times 25=125$	22.5	2812.5	63281.25	$\frac{1}{12}(25)(5)^3=260.42$
2	$5\times 20=100$	10	1000	10000	$\frac{1}{12}(5)(20)^3=3333.33$
3	$-\pi\left(\frac{4}{2}\right)^2=-12.57$	22.5	-282.83	-6363.68	$-\pi\left(\frac{4}{2}\right)^4=-50.27$
	212.43		3529.67	66917.57	3543.48

$$A=\sum A_i=212.43\,cm^2 \qquad \bar{y}=\frac{\sum A_i\bar{y}_i}{\sum A_i}=\frac{3529.67}{212.43}=16.62\,cm$$

$$I_{R-R}=\sum I_{g\,i}+\sum A_i\bar{y}_i^{\,2}=3543.48+66917.57=70461.05\,cm^4$$

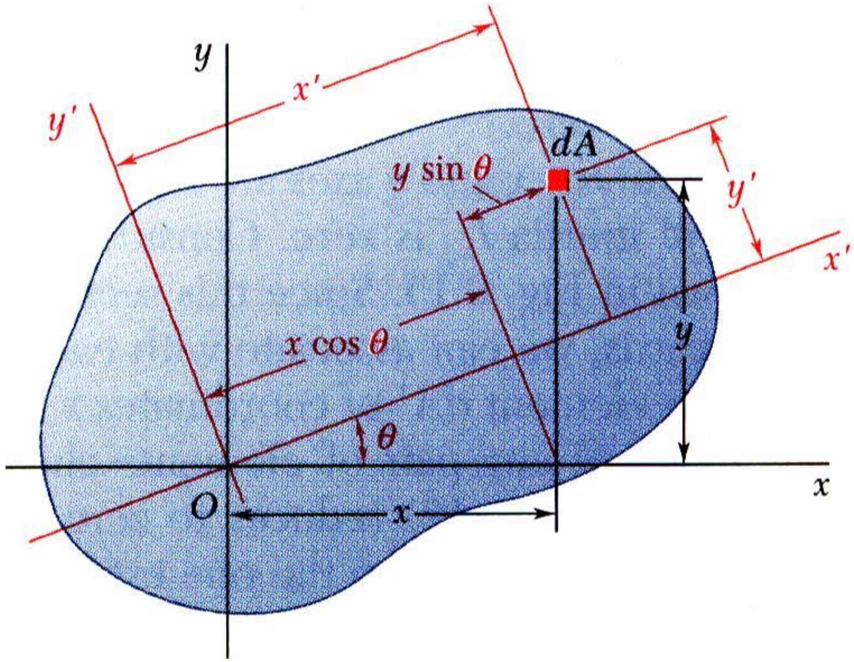
$$I_{NA}=\sum I_{g\,i}+\sum A_i\bar{y}_i^{\,2}-\frac{(\sum A_i\bar{y}_i)^2}{\sum A_i}=3543.48+66917.57-\frac{(3529.67)^2}{212.43}=11813.16\,cm^4$$

Principal Axes and Principal Moments of Inertia

Given:

$$I_x = \int y^2 dA \quad , \quad I_y = \int x^2 dA \quad , \quad I_{xy} = \int xy\, dA$$

we wish to determine moments and product of inertia with respect to new axes *x'* and *y'*.



Note:
$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

$$I_{x'} = \int (y')^2\ dA = \int (y\cos\theta - x\sin\theta)^2\ dA$$

$$= \cos^2\theta \int y^2\ dA - 2\sin\theta\cos\theta \int xy\ dA + \sin^2\theta \int x^2\ dA \quad \Rightarrow$$

$$I_{x'} = I_x\cos^2\theta - 2I_{xy}\sin\theta\cos\theta + I_y\sin^2\theta$$

Principal Axes and Principal Moments of Inertia

Similarly:

$$I_{y'} = I_x \sin^2\theta + 2I_{xy}\sin\theta\cos\theta + I_y\cos^2\theta$$

$$I_{x'y'} = (I_x - I_y)\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta)$$

Recalling the trigonometric relations

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

- The change of axes yields

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad (I)$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{x'} + I_{y'} = I_x + I_y$$

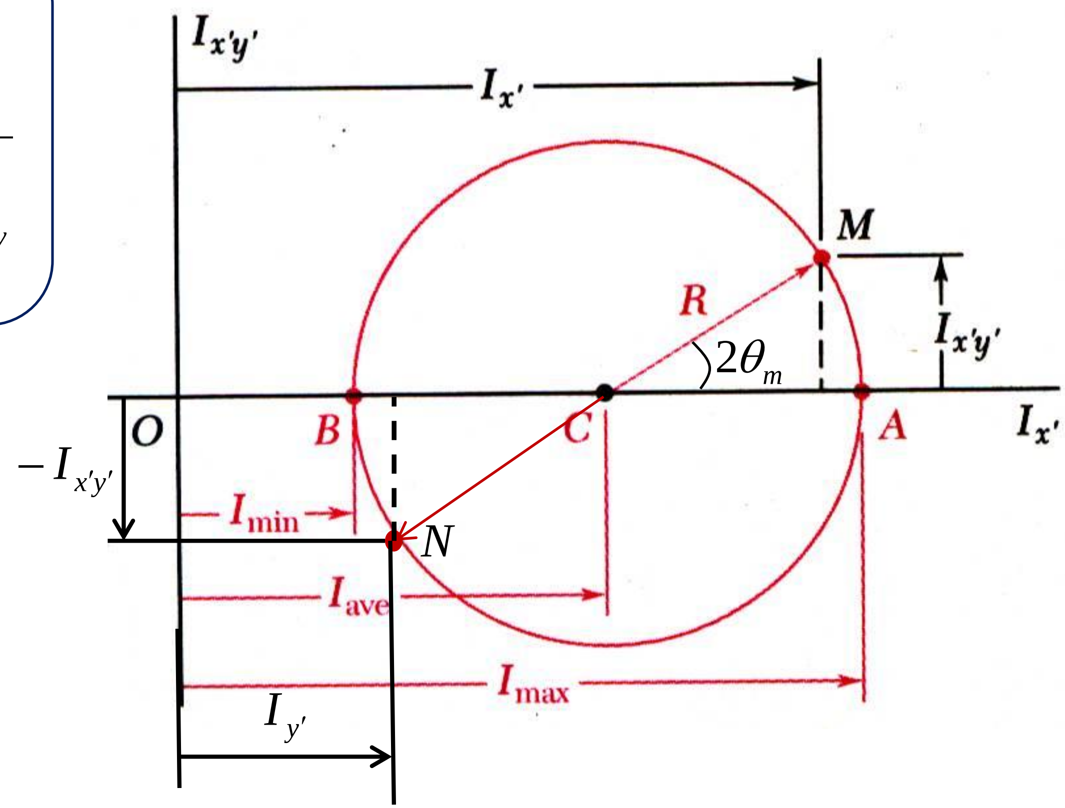
Principal Axes and Principal Moments of Inertia

we eliminate θ from Eqs. (I)

- The equations for $I_{x'}$ and $I_{x'y'}$ are the parametric equations for a circle,

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \quad , \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$



Principal Axes and Principal Moments of Inertia

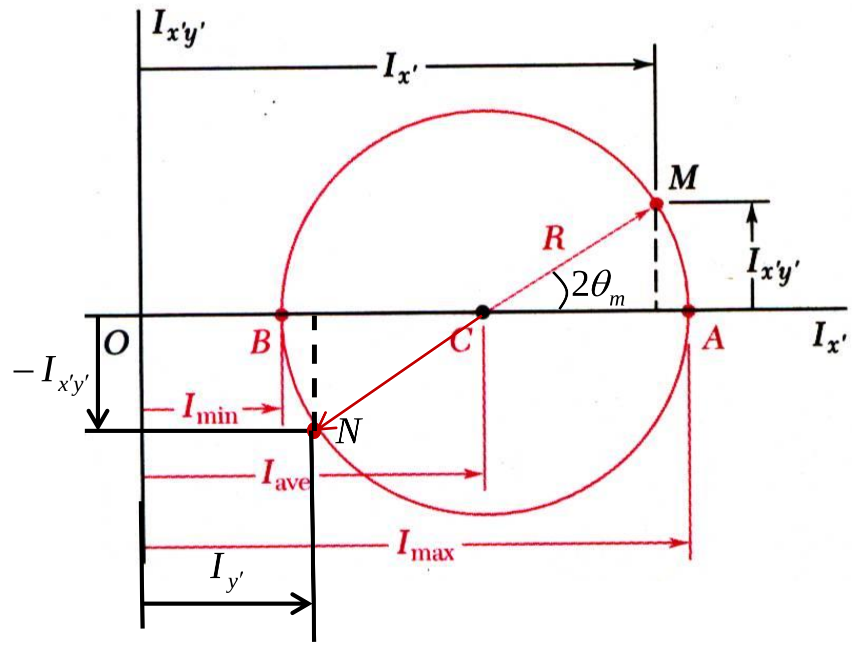
- At the points A and B, $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively.

$$I_{\max,\min} = I_{ave} \pm R \quad \Rightarrow$$

$$\boxed{I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}}$$

$$\boxed{\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}}$$

- The equation for θ_m defines two angles, 90° apart which correspond to the *principal axes* of the area about *O*.
- I_{max} and I_{min} are the *principal moments of inertia* of the area about *O*.

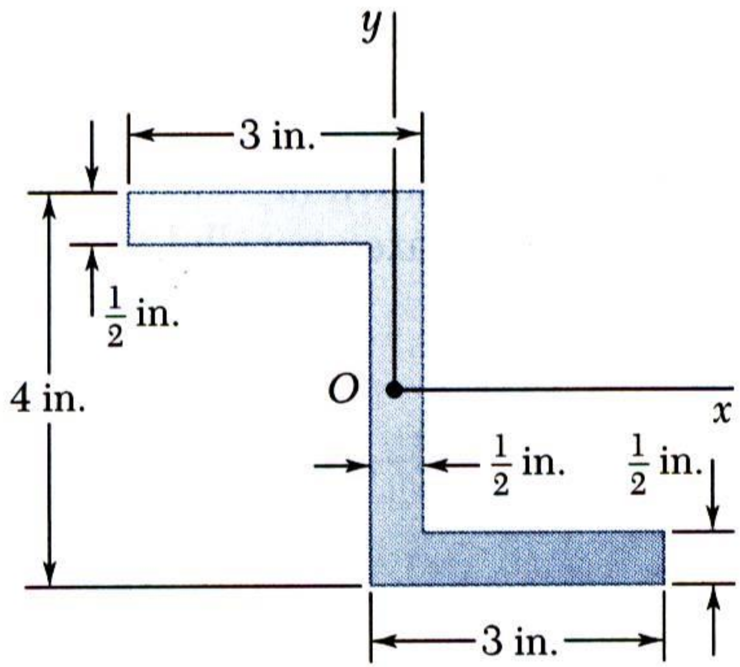


We note that if an area possesses an axis of symmetry through a point O , this axis must be a principal axis of the area about O. On the other hand, a principal axis does not need to be an axis of symmetry; weather or not an area possesses any axes of symmetry, it will have two principal axes of inertia about any point O.

❑ Sample Problem 08

For the section shown, the moments of inertia with respect to the x and y axes are $I_x = 10.38 \text{ in}^4$ and $I_y = 6.97 \text{ in}^4$.

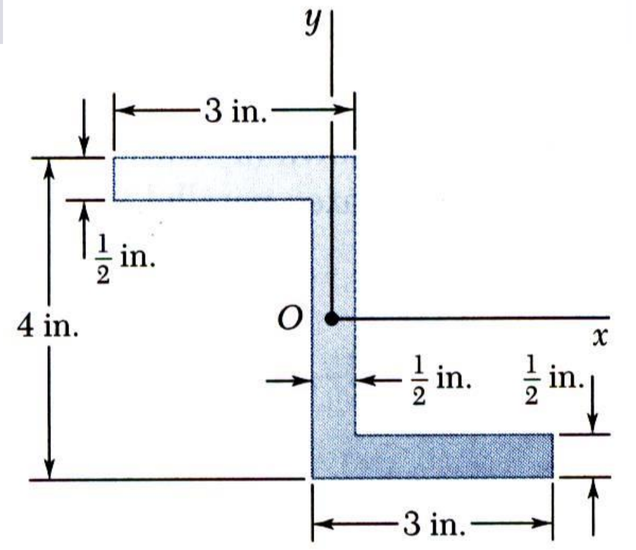
Determine *(a)* the orientation of the principal axes of the section about O, and *(b)* the values of the principal moments of inertia about O.



Sample Problem 08

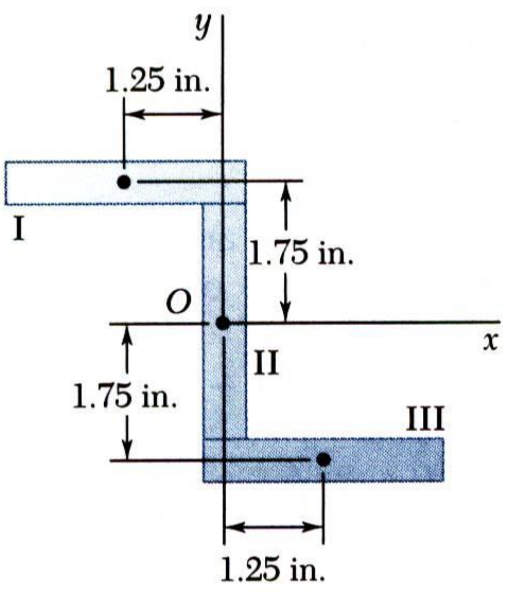
SOLUTION:

- Compute the product of inertia with respect to the *xy* axes by dividing the section into three rectangles.



Apply the parallel axis theorem to each rectangle,

$$I_{xy} = \sum \left(\bar{I}_{x'y'} + \bar{x}\bar{y}A \right)$$



Note that the product of inertia with respect to centroidal axes parallel to the *xy* axes is zero for each rectangle.

$$\bar{I}_{x'y'} = 0$$

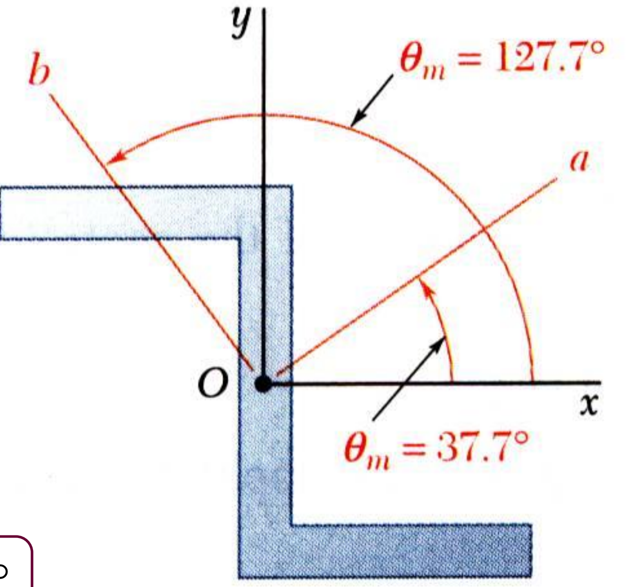
Rectangle	Area, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$, in^4
I	$3 \times \frac{1}{2} = 1.5$	-1.25	$+1.75$	-3.28
II	$3 \times \frac{1}{2} = 1.5$	0	0	0
III	$3 \times \frac{1}{2} = 1.5$	$+1.25$	-1.75	-3.28
				$\sum \bar{x}\bar{y}A = -6.56$

$$\Rightarrow \quad I_{xy} = \sum \bar{x}\bar{y}A = -6.56 \,(in^4)$$

❑ Sample Problem 08

SOLUTION:

- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9. 27).



$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

$$\Rightarrow \quad 2\theta_m = 75.4° \text{ and } 255.4°$$

$$\Rightarrow \boxed{\theta_m = 37.7° \ , \ \theta_m = 127.7°}$$

$$I_x = 10.38 \text{ in}^4$$
$$I_y = 6.97 \text{ in}^4$$
$$I_{xy} = -6.56 \text{ in}^4$$

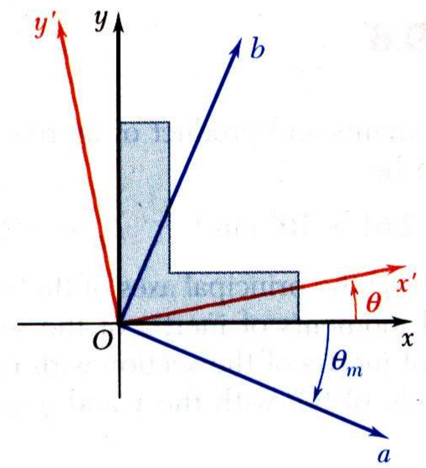
$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2}$$

$$\Rightarrow \boxed{\begin{aligned} I_a &= I_{\max} = 15.45\,(\text{in}^4) \\ I_b &= I_{\min} = 1.897\,(\text{in}^4) \end{aligned}}$$

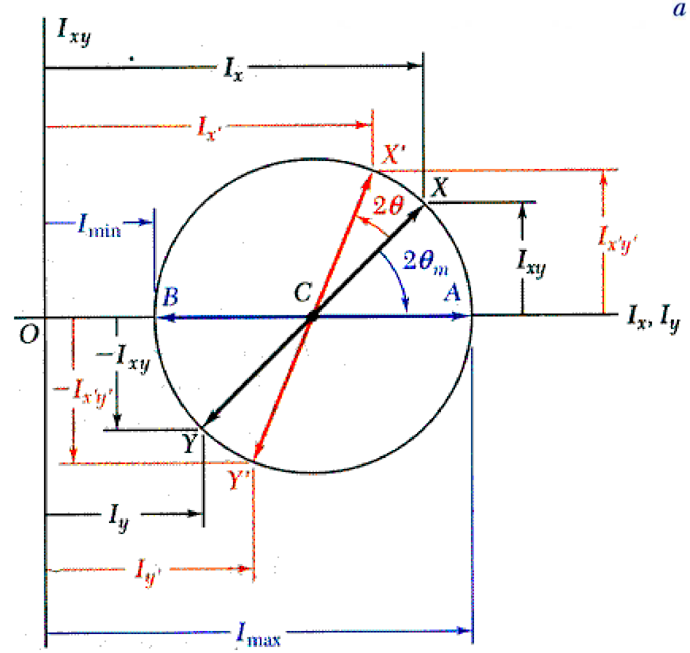
Mohr's Circle for Moments and Products of Inertia

Introduced by the German engineer **Otto Mohr** (1835-1918) and is known as **Mohr's circle**.



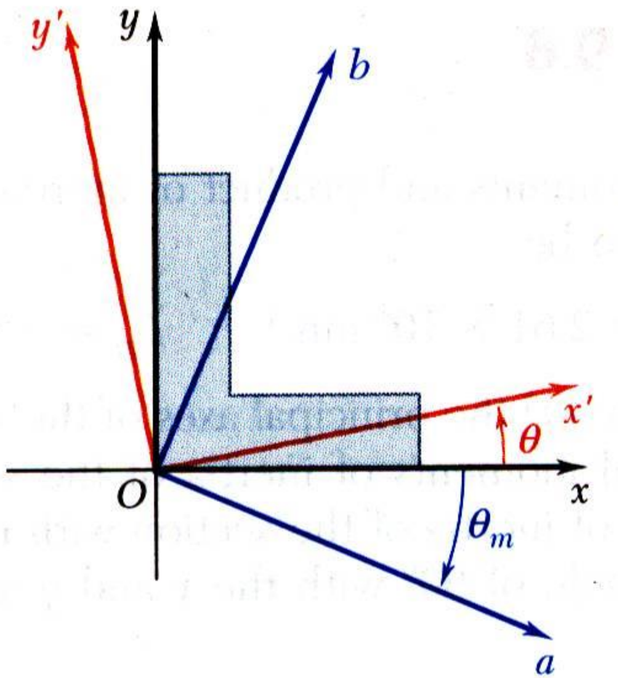
- The moments and product of inertia for an area are plotted as shown and used to construct *Mohr's circle,*

- Mohr's circle may be used to graphically or analytically determine the moments and product of inertia for any other rectangular axes including the principal axes and principal moments and products of inertia.



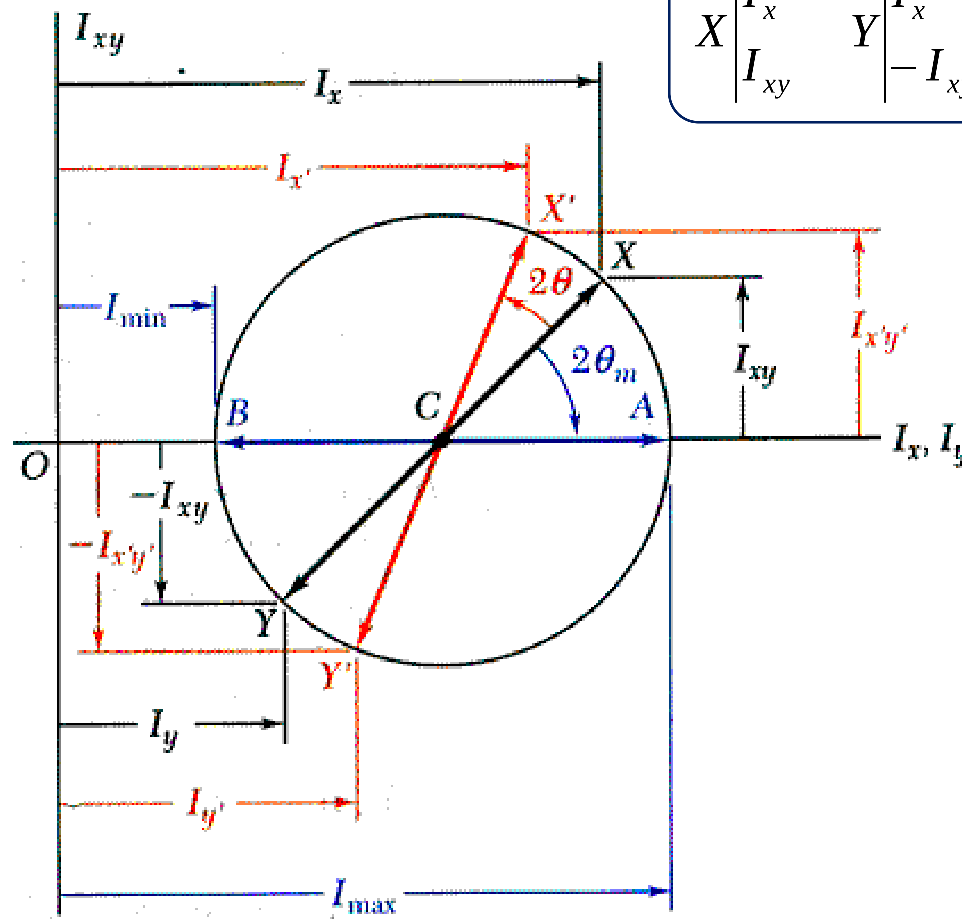
Mohr’s Circle for Moments and Products of Inertia

$$X\left| \begin{matrix} I_x \\ I_{xy} \end{matrix} \right. \qquad Y\left| \begin{matrix} I_x \\ -I_{xy} \end{matrix} \right.$$



$I_x\ ,\ I_y\ ,\ I_{xy}$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right) + I_{xy}^2}$$

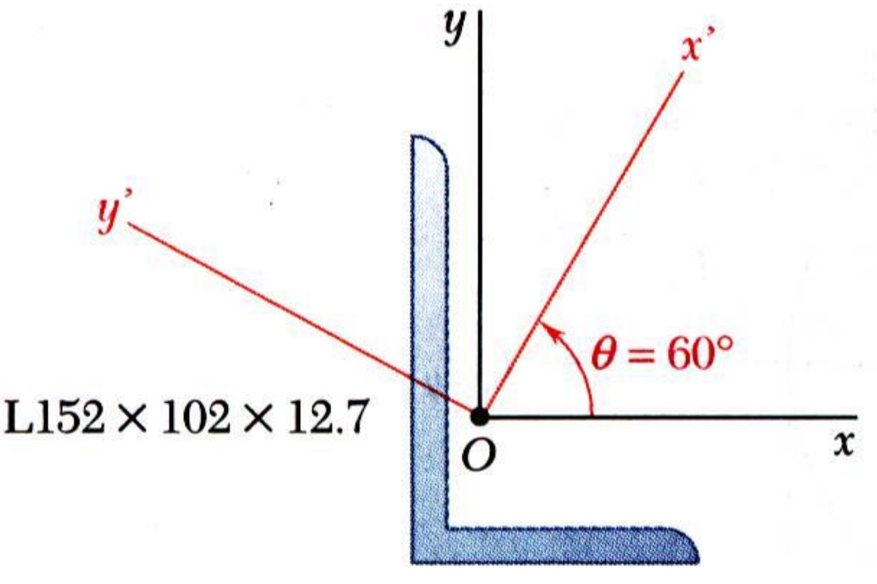


❑ Sample Problem 09

The moments and product of inertia with respect to the *x* and *y* axes are

$I_x = 7.24\text{x}10^6\ \text{mm}^4$, $I_y = 2.61\text{x}10^6\ \text{mm}^4$, and $I_{xy} = -2.54\text{x}10^6\ \text{mm}^4$.

Using Mohr's circle, determine *(a)* the principal axes about *O*, *(b)* the values of the principal moments about *O*, and *(c)* the values of the moments and product of inertia about the *x'* and *y'* axes

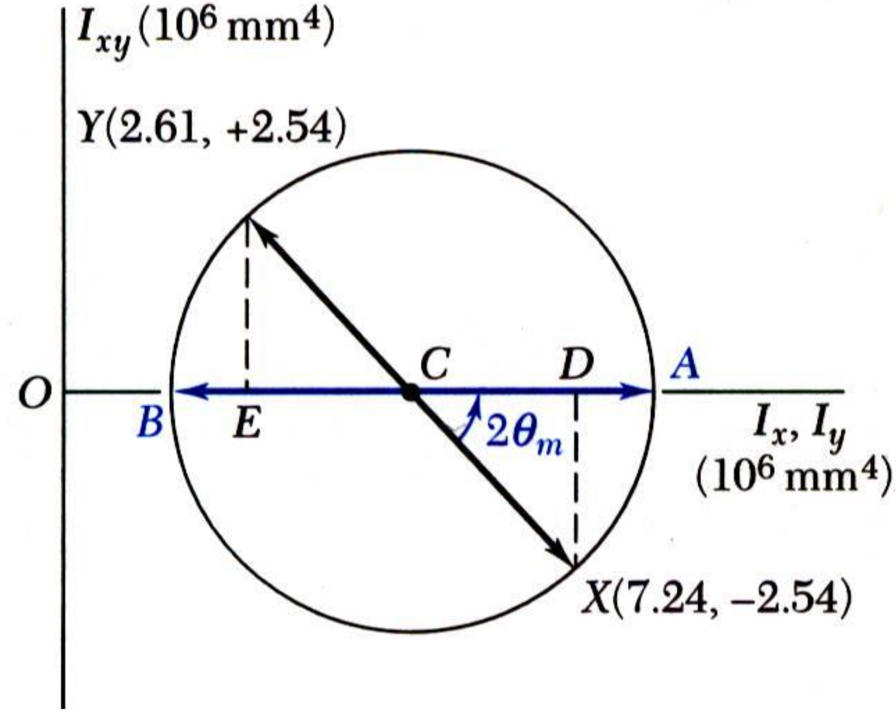


Sample Problem 09

SOLUTION:

- Plot the points (I_x , I_{xy}) and (I_y ,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.

$$I_x = 7.24\times10^6\,\text{mm}^4$$
$$I_y = 2.61\times10^6\,\text{mm}^4$$
$$I_{xy} = -2.54\times10^6\,\text{mm}^4$$



$$OC = I_{ave} = \frac{1}{2}(I_x + I_y) = \frac{1}{2}(7.24\times10^6 + 2.61\times10^6) \Rightarrow \boxed{OC = 4.925\times10^6(\text{mm}^4)}$$

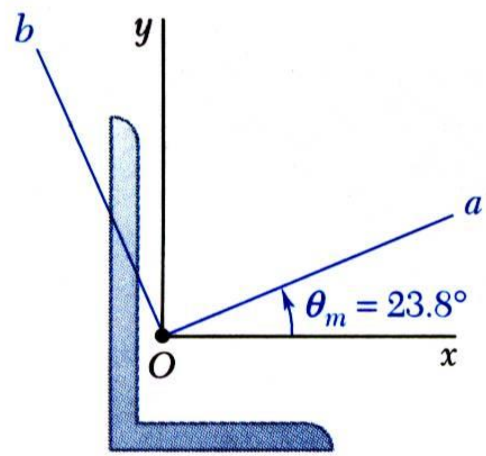
$$CD = \frac{1}{2}(I_x - I_y) = \frac{1}{2}(7.24\times10^6 - 2.61\times10^6) \Rightarrow \boxed{CD = 2.315\times10^6\ (\text{mm}^4)}$$

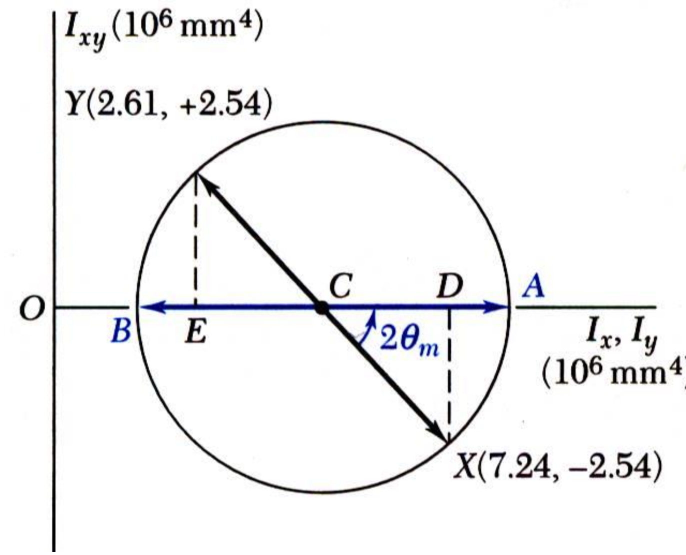
$$R = \sqrt{(CD)^2 + (DX)^2} = \sqrt{(2.315\times10^6)^2 + (-2.54\times10^6)^2} \Rightarrow \boxed{R = 3.437\times10^6\ (\text{mm}^4)}$$

Sample Problem 09

SOLUTION:

- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.





$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad \Rightarrow \quad 2\theta_m = 47.6° \quad \Rightarrow \quad \boxed{\theta_m = 23.8°}$$

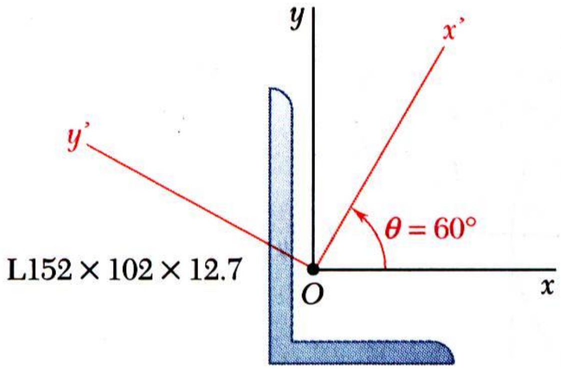
$$I_{\max} = OA = I_{ave} + R = 4.925\times10^6 + 3.437\times10^6 \Rightarrow \boxed{I_{\max} = 8.36\times10^6 (\mathrm{mm}^4)}$$

$$I_{\min} = OB = I_{ave} - R = 4.925\times10^6 - 3.437\times10^6 \Rightarrow \boxed{I_{\min} = 1.49\times10^6 (\mathrm{mm}^4)}$$

❑ Sample Problem 09

SOLUTION:

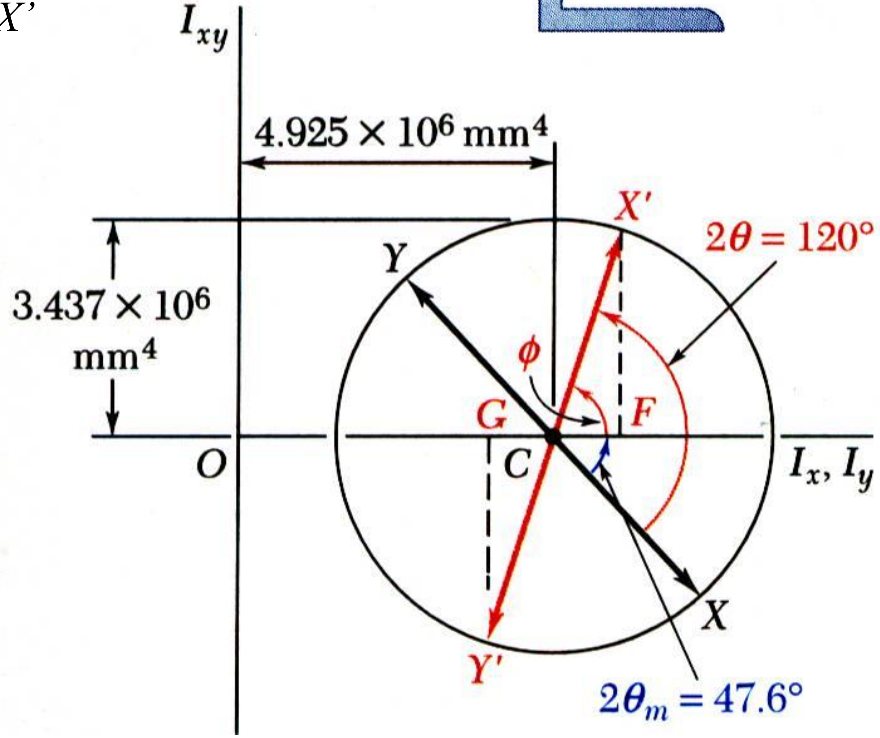
The points X' and Y' corresponding to the x' and y' axes are obtained by rotating CX and CY counterclockwise through an angle $\theta = 2(60^o) = 120^o$. The angle that CX' forms with the x' axes is $\phi = 120^o - 47.6^o = 72.4^o$.



$$I_{x'} = OF = OC + CX' \cos\varphi = I_{ave} + R\cos 72.4^o$$

$$= 4.925\times10^6 + (3.437\times10^6)\cos 72.4^o$$

$$\Rightarrow \boxed{I_{x'} = 5.96\times10^6 (\text{mm}^4)}$$



$$I_{y'} = OG = OC - CY' \cos\varphi = I_{ave} - R\cos 72.4^o$$

$$= 4.925\times10^6 - (3.437\times10^6)\cos 72.4^o$$

$$\Rightarrow \boxed{I_{y'} = 3.89\times10^6 (\text{mm}^4)}$$

$$I_{x'y'} = FX' = CY' \sin\varphi = R\sin 72.4^o = (3.437\times10^6)\sin 72.4^o$$

$$\Rightarrow \boxed{I_{x'y'} = 3.28\times10^6 (\text{mm}^4)}$$

$$\boxed{\begin{aligned} OC &= I_{ave} = 4.925\times10^6 \,\text{mm}^4 \\ R &= 3.437\times10^6 \,\text{mm}^4 \end{aligned}}$$