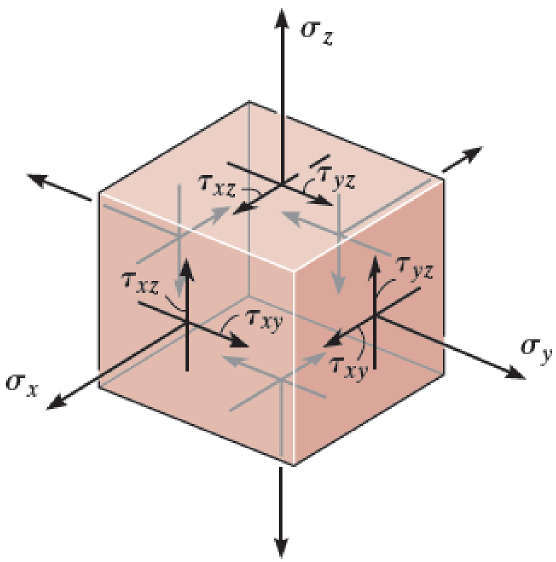
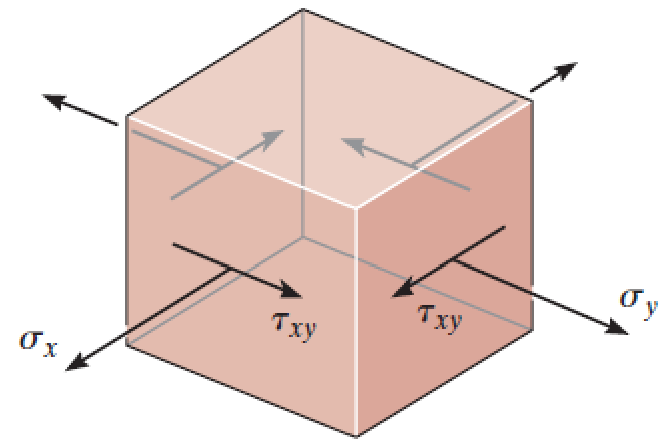
STRESS TRANSFORMATION

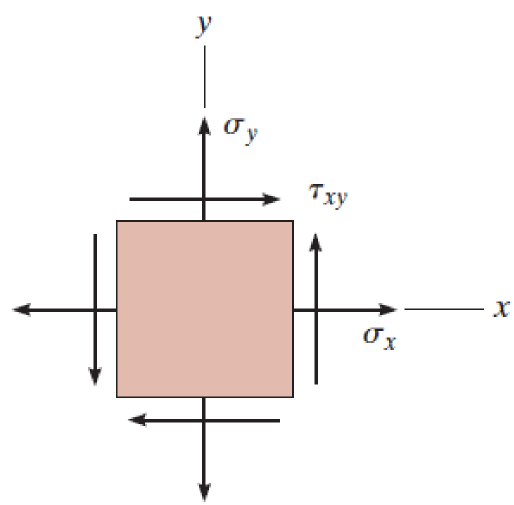
فصل هشتم: تبدیل تنش

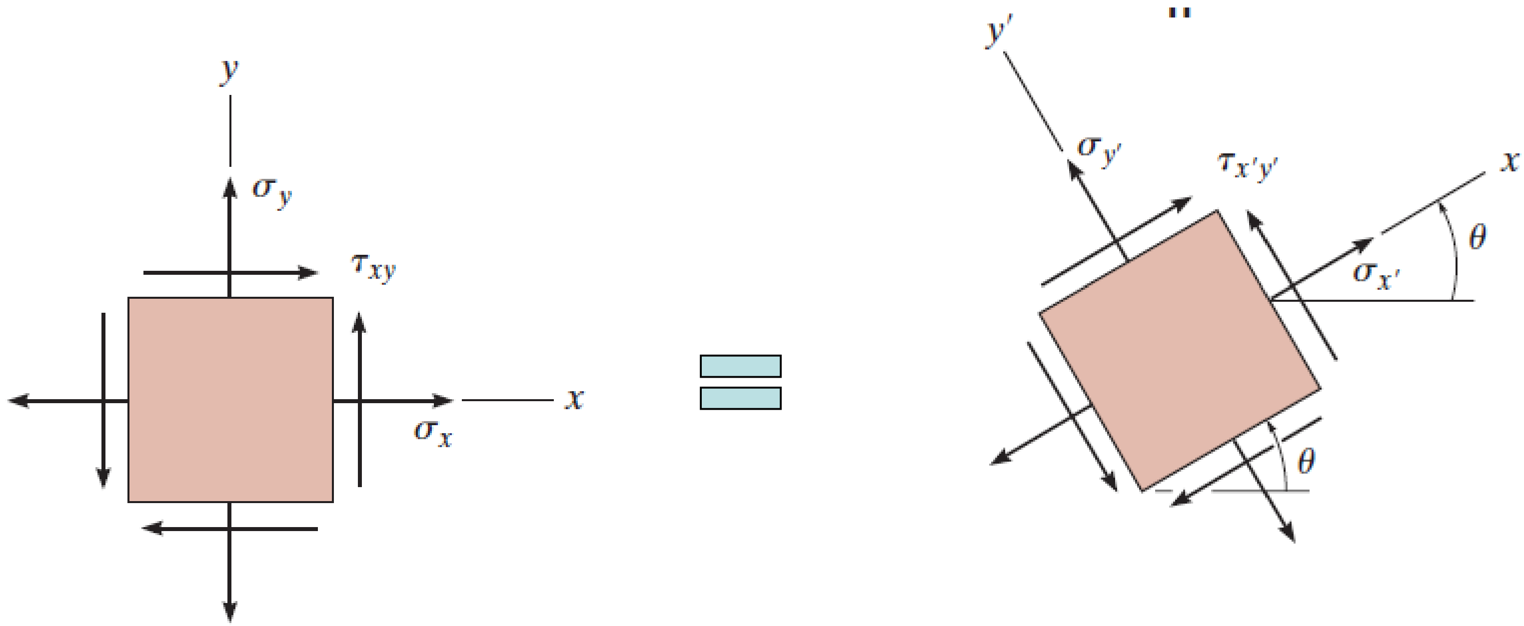


General state of stress



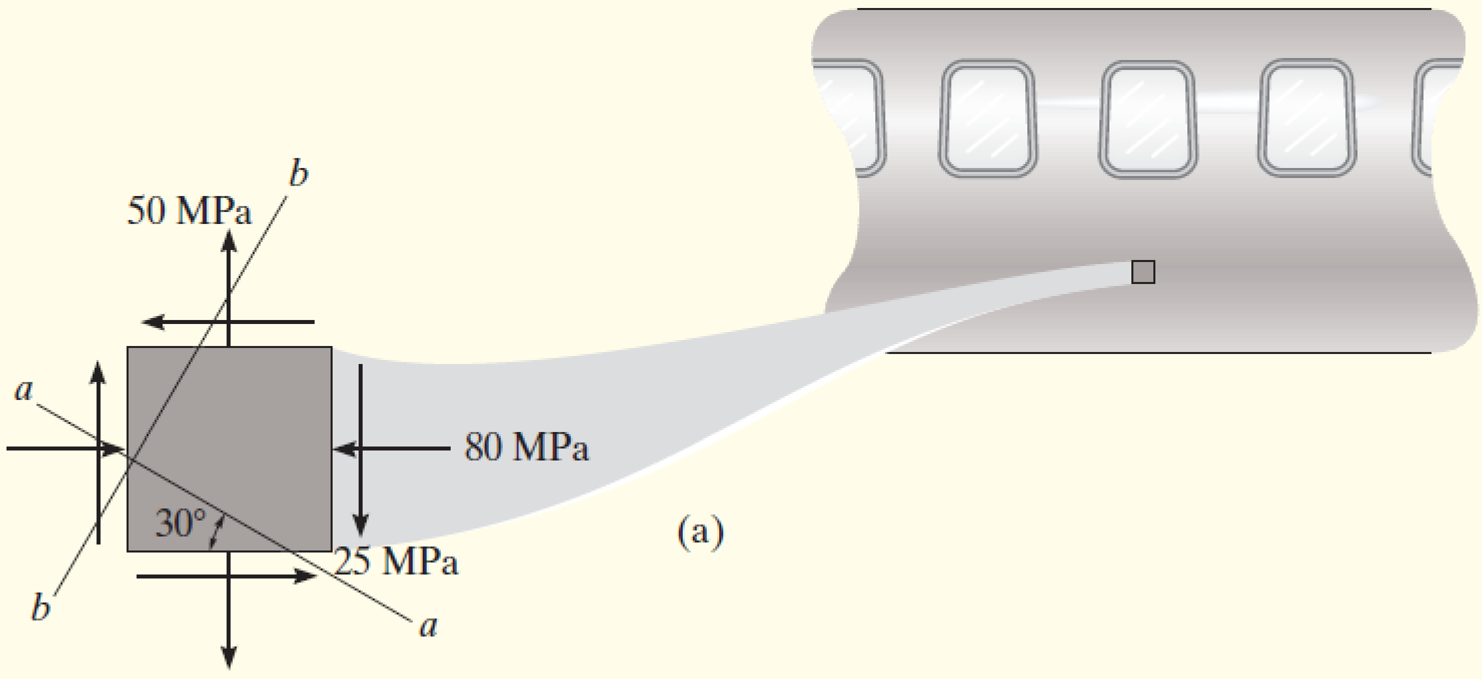
Plane stress




y
σy
τxy
x
σx
y′
σy′
τx′y′
x′
θ
σx′
θ

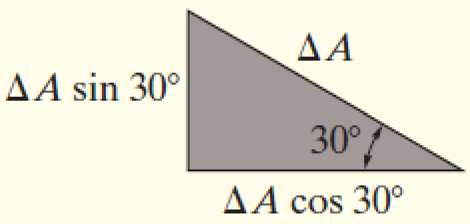
Example

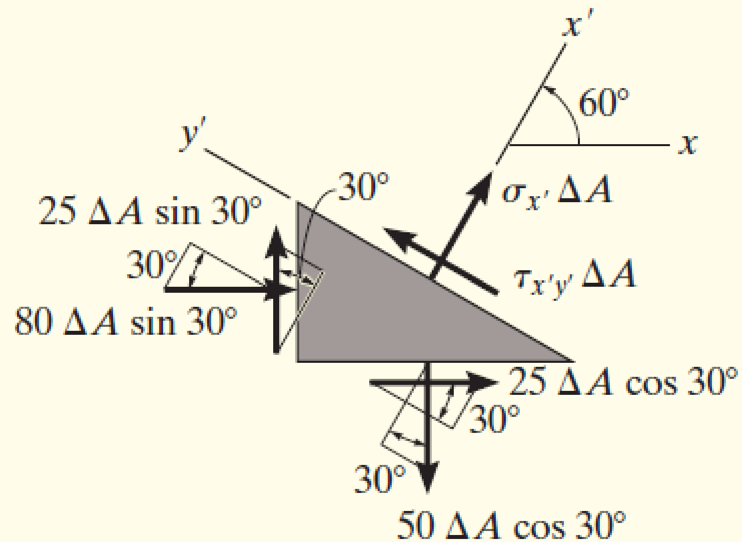
The state of plane stress at a point on the surface of the airplane fuselage is represented on the element oriented as shown in Fig. 9–4*a*. Represent the state of stress at the point on an element that is oriented 30° clockwise from this position.



(a)

Example





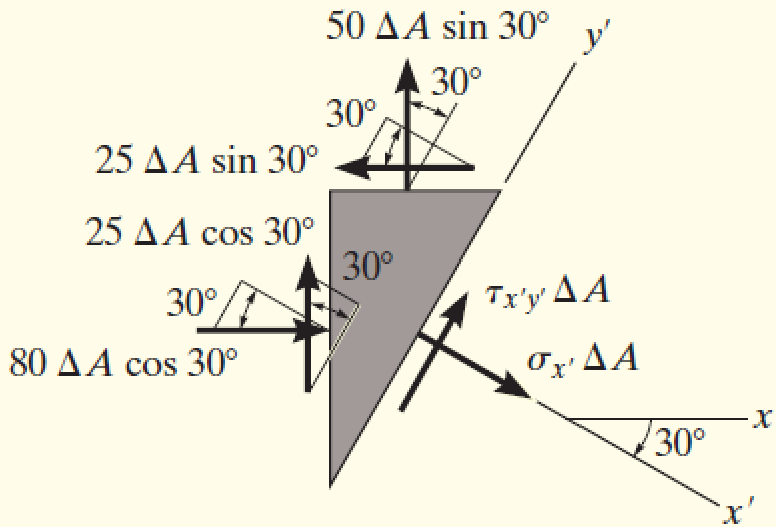
Equilibrium.

$$+\nearrow \Sigma F_{x'} = 0; \qquad \sigma_{x'}\Delta A - (50\ \Delta A \cos 30^\circ)\cos 30^\circ + (25\ \Delta A \cos 30^\circ)\sin 30^\circ + (80\ \Delta A \sin 30^\circ)\sin 30^\circ + (25\ \Delta A \sin 30^\circ)\cos 30^\circ = 0$$

$$\sigma_x{}' = -4.15\text{ MPa} \qquad \textit{Ans.}$$

$$+\nwarrow \Sigma F_{y'} = 0; \qquad \tau_{x'y'}\Delta A - (50\ \Delta A \cos 30^\circ)\sin 30^\circ - (25\ \Delta A \cos 30^\circ)\cos 30^\circ - (80\ \Delta A \sin 30^\circ)\cos 30^\circ + (25\ \Delta A \sin 30^\circ)\sin 30^\circ = 0$$

$$\tau_{x'y'} = 68.8\text{ MPa} \qquad \textit{Ans.}$$



Equilibrium.

$$+\searrow\Sigma F_{x'} = 0; \quad \sigma_{x'}\Delta A - (25\,\Delta A\cos 30^\circ)\sin 30^\circ + (80\,\Delta A\cos 30^\circ)\cos 30^\circ - (25\,\Delta A\sin 30^\circ)\cos 30^\circ - (50\,\Delta A\sin 30^\circ)\sin 30^\circ = 0$$

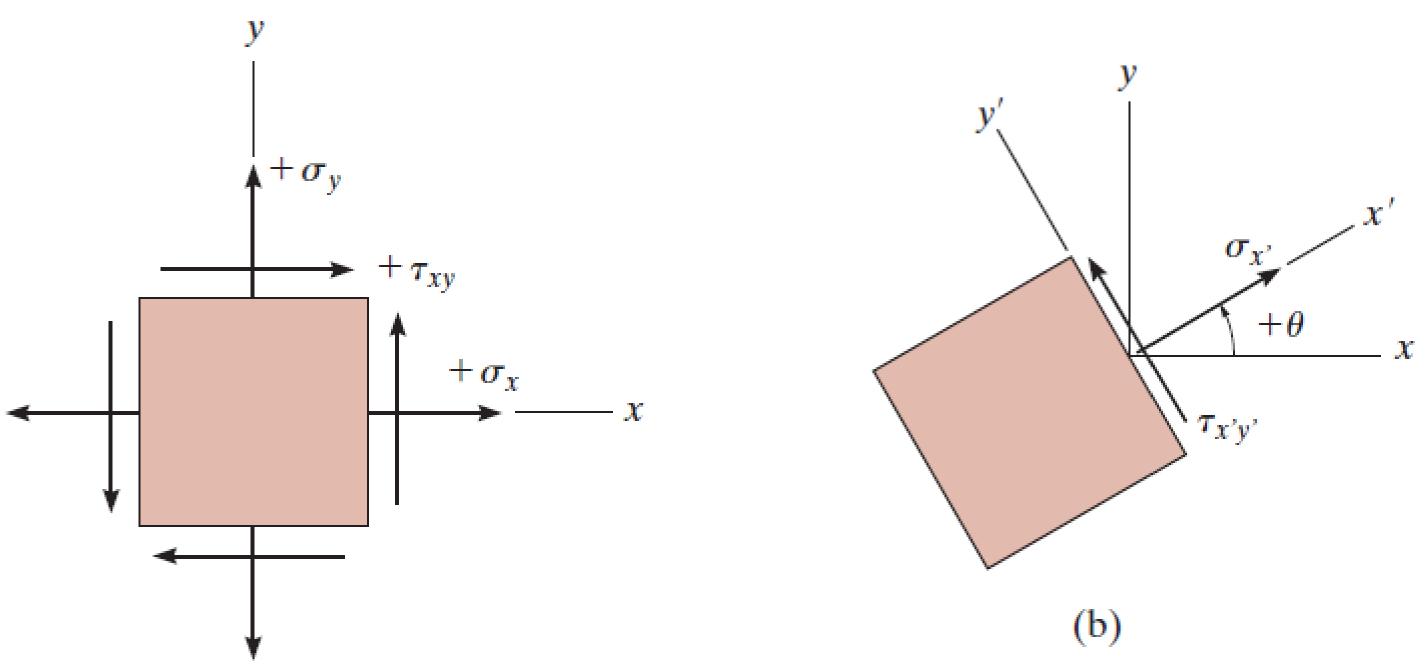
$$\sigma_{x'} = -25.8\text{ MPa} \qquad \textit{Ans.}$$

$$+\nearrow\Sigma F_{y'} = 0; \quad \tau_{x'y'}\,\Delta A + (25\,\Delta A\cos 30^\circ)\cos 30^\circ + (80\,\Delta A\cos 30^\circ)\sin 30^\circ - (25\,\Delta A\sin 30^\circ)\sin 30^\circ + (50\,\Delta A\sin 30^\circ)\cos 30^\circ = 0$$

$$\tau_{x'y'} = -68.8\text{ MPa} \qquad \textit{Ans.}$$

GENERAL EQUATIONS OF PLANE-STRESS TRANSFORMATION

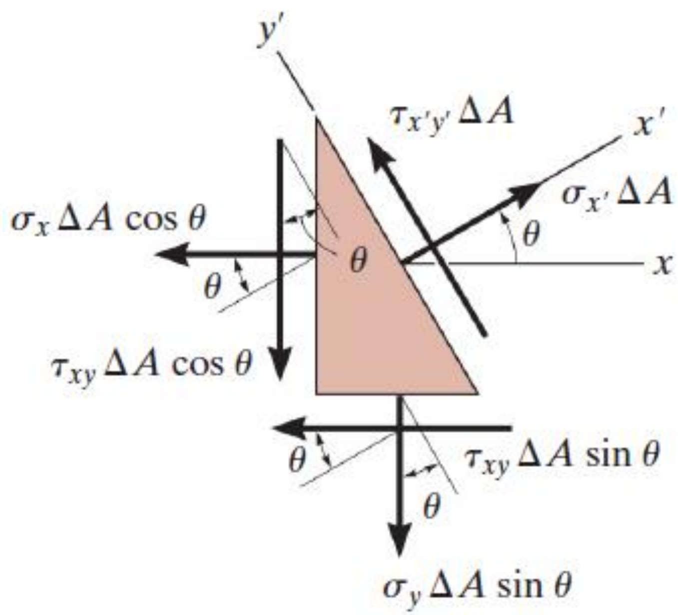
Sign Convention.

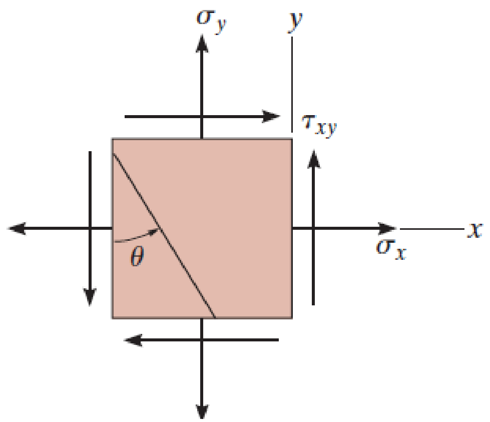


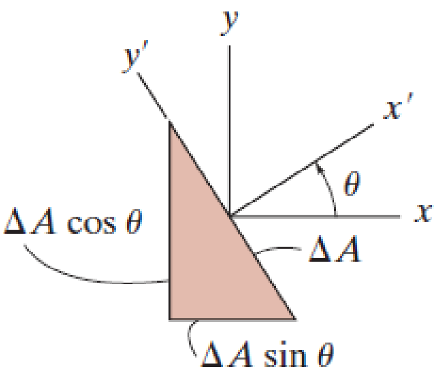
(b)

Positive sign convention

Normal and Shear Stress Components.





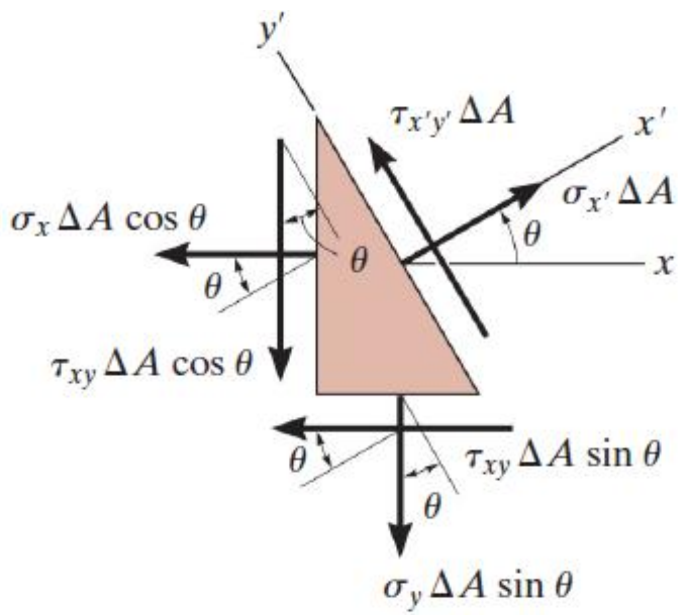


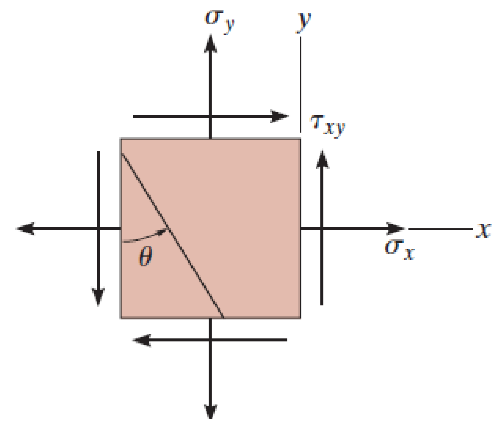
$$+\nearrow \Sigma F_{x'} = 0; \quad \sigma_{x'}\Delta A - (\tau_{xy}\Delta A \sin\theta)\cos\theta - (\sigma_y \Delta A \sin\theta)\sin\theta$$

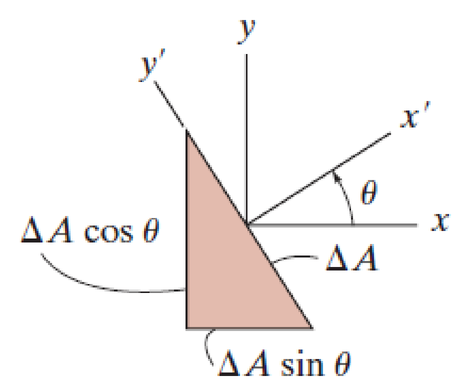
$$- (\tau_{xy}\Delta A \cos\theta)\sin\theta - (\sigma_x \Delta A \cos\theta)\cos\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

Normal and Shear Stress Components.







$$+\nwarrow \Sigma F_{y'} = 0; \quad \tau_{x'y'}\Delta A + (\tau_{xy}\Delta A \sin\theta)\sin\theta - (\sigma_y \Delta A \sin\theta)\cos\theta$$

$$- (\tau_{xy}\Delta A \cos\theta)\cos\theta + (\sigma_x \Delta A \cos\theta)\sin\theta = 0$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Normal and Shear Stress Components.

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

To simplify these two equations, use the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin^2\theta = (1 - \cos 2\theta)/2$, and $\cos^2\theta = (1 + \cos 2\theta)/2$. Therefore,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

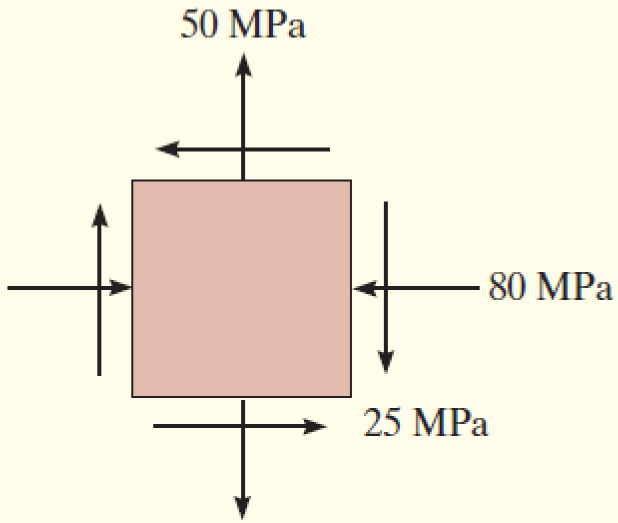
Example

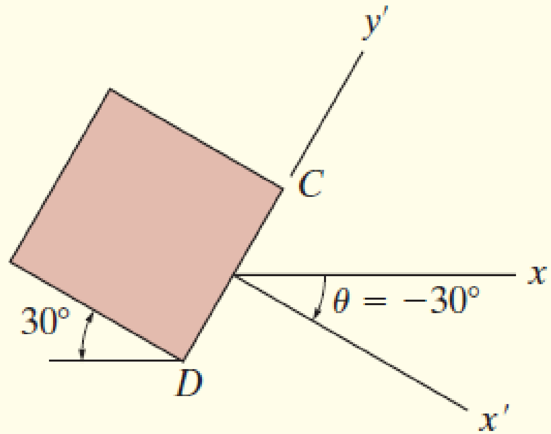
The state of plane stress at a point is represented on the element shown in Fig. 9–7*a*. Determine the state of stress at this point on another element oriented 30° clockwise from the position shown.

SOLUTION

This problem was solved in Example 9.1 using basic principles. Here we will apply Eqs. 9–1 and 9–2. From the established sign convention, Fig. 9–5, it is seen that

$$\sigma_x = -80 \text{ MPa} \qquad \sigma_y = 50 \text{ MPa} \qquad \tau_{xy} = -25 \text{ MPa}$$





Plane *CD*.

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

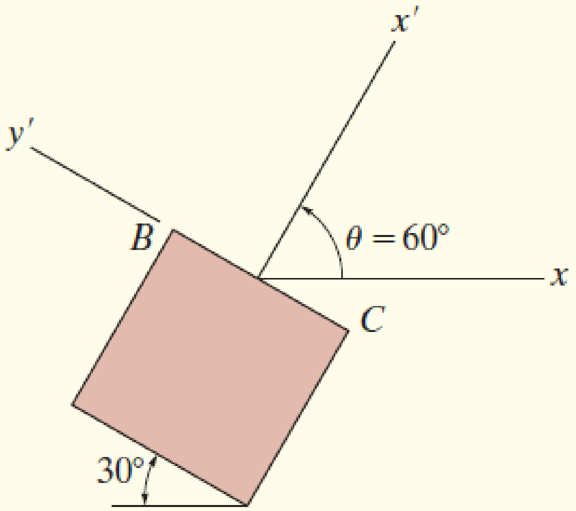
$$= \frac{-80 + 50}{2} + \frac{-80 - 50}{2}\cos 2(-30^\circ) + (-25)\sin 2(-30^\circ)$$

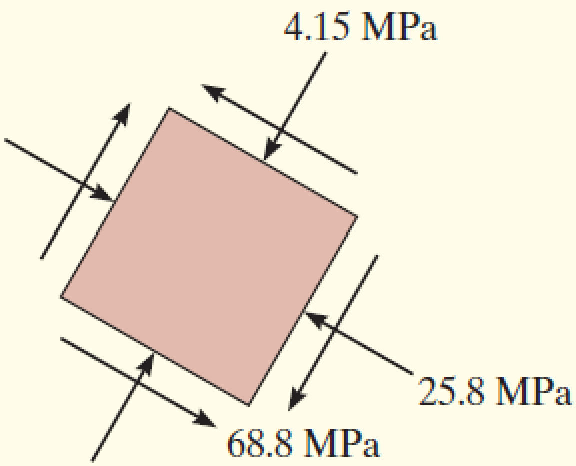
$$= -25.8\text{ MPa}$$ *Ans.*

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\frac{-80 - 50}{2}\sin 2(-30^\circ) + (-25)\cos 2(-30^\circ)$$

$$= -68.8\text{ MPa}$$ *Ans.*





Plane *BC*.

$$\sigma_{x'} = \frac{-80 + 50}{2} + \frac{-80 - 50}{2}\cos 2(60^\circ) + (-25)\sin 2(60^\circ)$$

$$= -4.15 \text{ MPa} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = -\frac{-80 - 50}{2}\sin 2(60^\circ) + (-25)\cos 2(60^\circ)$$

$$= 68.8 \text{ MPa} \qquad \textit{Ans.}$$

PRINCIPAL STRESSES AND MAXIMUM IN-PLANE SHEAR STRESS

In-Plane Principal Stresses. To determine the maximum and minimum *normal stress*, we must differentiate Eq. 9–1 with respect to θ and set the result equal to zero. This gives

$$\frac{d\sigma_{x'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2 \sin 2\theta) + 2\tau_{xy} \cos 2\theta = 0$$

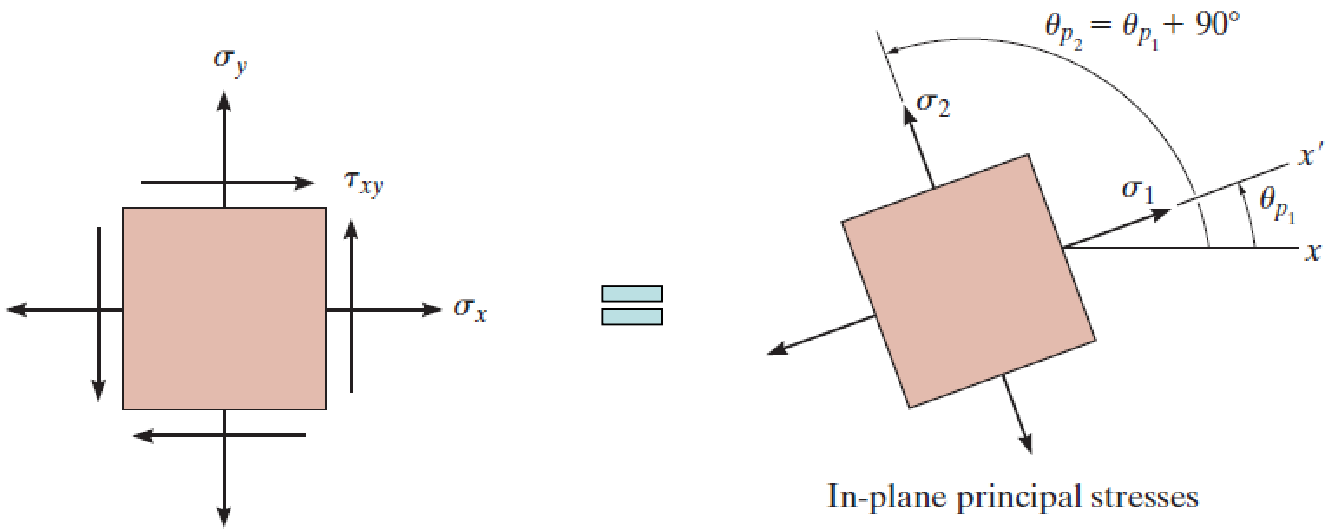
$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

Orientation of Principal Planes

principal stresses,

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

no shear stress acts on the principal planes,



In-plane principal stresses

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Maximum In-Plane Shear Stress. The orientation of the element that is subjected to maximum shear stress can be determined by taking the derivative of Eq. 9–2 with respect to θ, and setting the result equal to zero. This gives

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}} \qquad (9\text{–}6)$$

Orientation of Maximum In-Plane Shear Stress

$$\tau_{\substack{\max \\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Maximum In-Plane Shear Stress

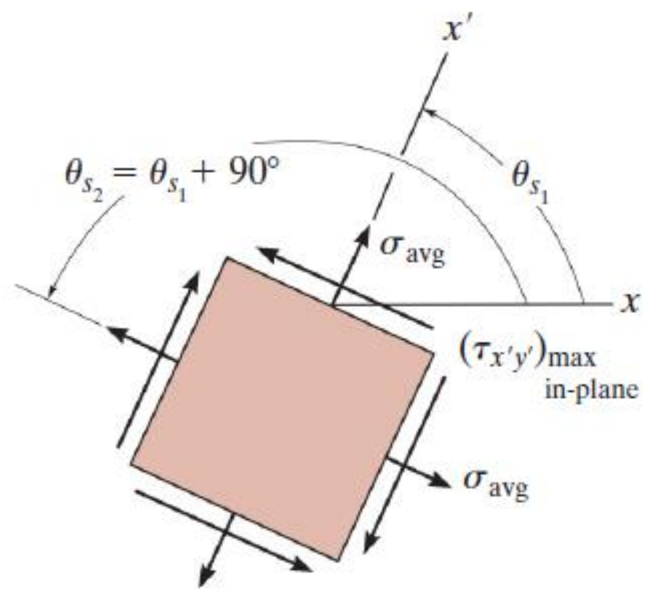
maximum shear stress must be oriented 45° from the position of an element that is subjected to the principal stress.

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Finally, when the values for $\sin 2\theta_s$ and $\cos 2\theta_s$ are substituted into Eq. 9–1, we see that there is *also* an ***average normal stress*** on the planes of maximum in-plane shear stress. It is

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

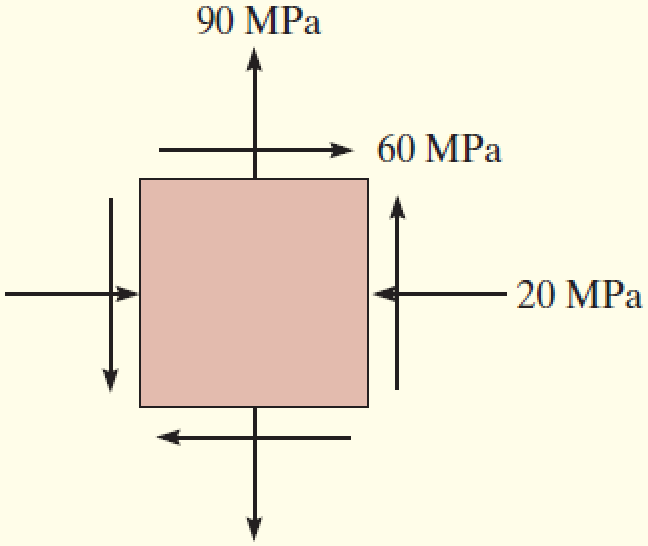
Average Normal Stress



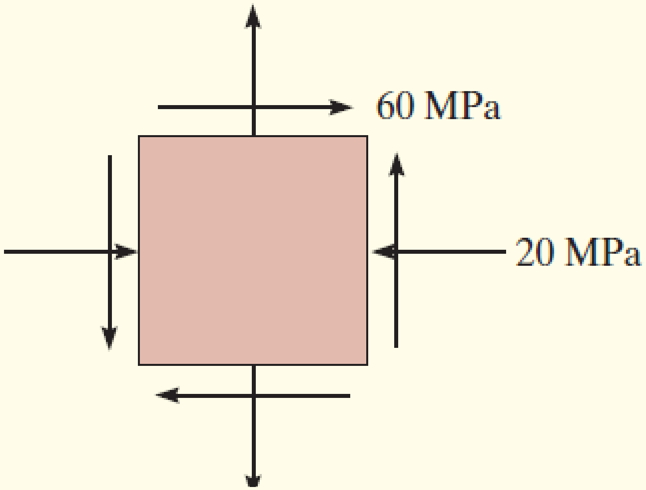
Maximum in-plane shear stresses

Example

The state of plane stress at a point on a body is represented on the element shown in Fig. 9–12*a*. Represent this state of stress in terms of its maximum in-plane shear stress and associated average normal stress.



Example



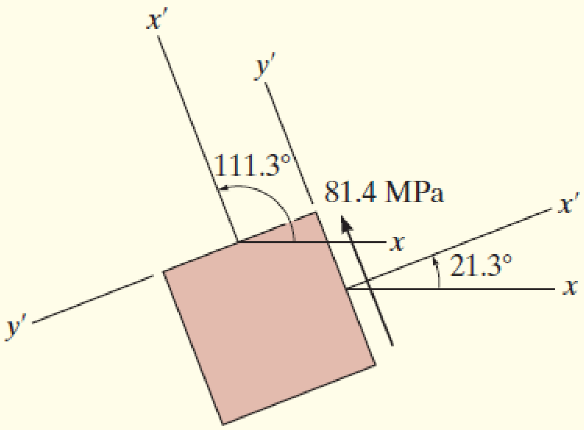
Orientation of Element. Since $\sigma_x = -20$ MPa, $\sigma_y = 90$ MPa, and $\tau_{xy} = 60$ MPa, applying Eq. 9–6, the two angles are

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}} = \frac{-(-20 - 90)/2}{60}$$

$$2\theta_{s_2} = 42.5^\circ \qquad \theta_{s_2} = 21.3^\circ$$

$$2\theta_{s_1} = 180^\circ + 2\theta_{s_2} \qquad \theta_{s_1} = 111.3^\circ$$

Example



Maximum In-Plane Shear Stress. Applying Eq. 9–7,

$$\tau_{\substack{\max\\ \text{in-plane}}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sqrt{\left(\frac{-20 - 90}{2}\right)^2 + (60)^2}$$

$$= \pm 81.4 \text{ MPa} \qquad \textit{Ans.}$$

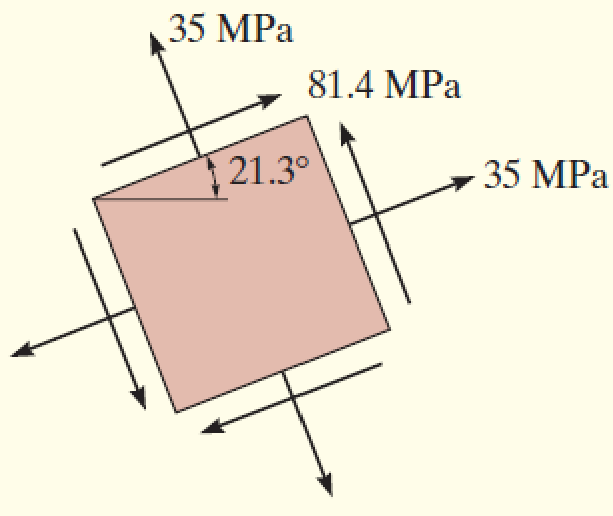
The proper direction of $\tau_{\substack{\max\\ \text{in-plane}}}$ on the element can be determined by substituting $\theta = \theta_{s_2} = 21.3^\circ$ into Eq. 9–2. We have

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\left(\frac{-20 - 90}{2}\right)\sin 2(21.3^\circ) + 60\cos 2(21.3^\circ)$$

$$= 81.4 \text{ MPa}$$

Example

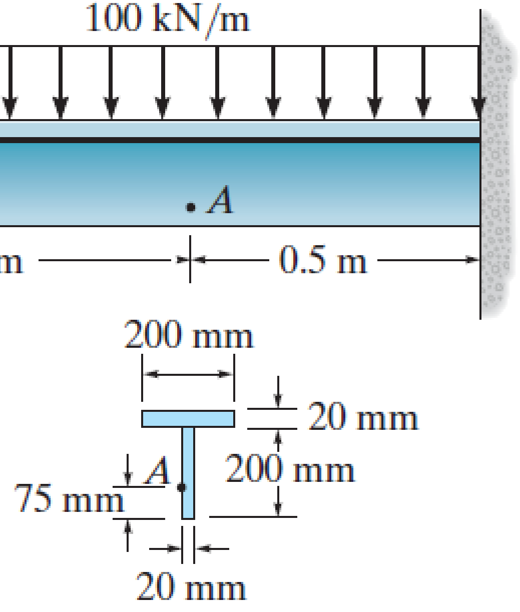


Average Normal Stress. Besides the maximum shear stress, the element is also subjected to an average normal stress determined from Eq. 9–8; that is,

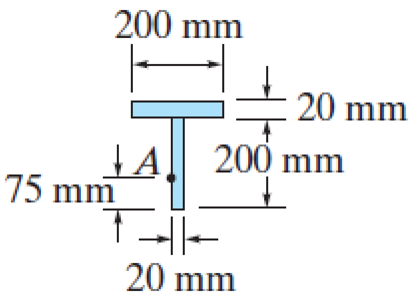
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-20 + 90}{2} = 35 \text{ MPa} \qquad \textit{Ans.}$$

Example

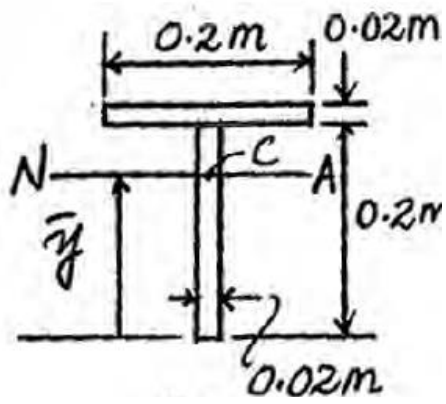
The T-beam is subjected to the distributed loading that is applied along its centerline. Determine the principal stress at point *A* and show the results on an element located at this point.



Example



The location of the centroid c of the T cross-section, Fig. a, is



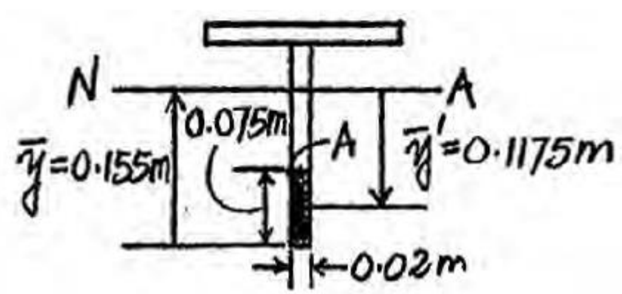
$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A} = \frac{0.1(0.2)(0.02) + 0.21(0.02)(0.2)}{0.2(0.02) + 0.02(0.2)} = 0.155 \text{ m}$$

$$I = \frac{1}{12}(0.02)(0.2^3) + 0.02(0.2)(0.155 - 0.1)^2$$

$$+ \frac{1}{12}(0.2)(0.02^3) + 0.2(0.02)(0.21 - 0.155)^2$$

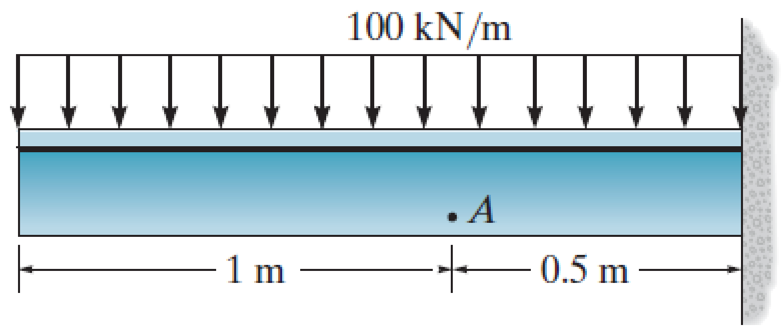
$$= 37.6667(10^{-6}) \text{ m}^4$$

Referring to Fig. b,



$$Q_A = \bar{y}' A' = 0.1175(0.075)(0.02) = 0.17625(10^{-3}) \text{ m}^3$$

Example



Using the method of sections and considering the FBD of the left cut segment of the beam, Fig. *c*,

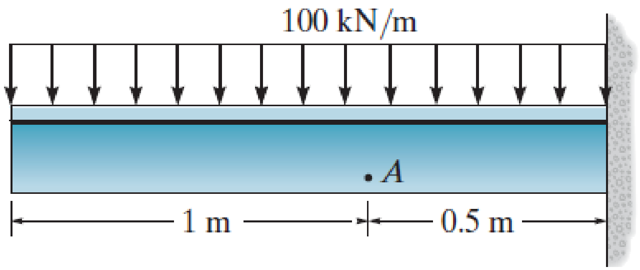
$$+\uparrow \Sigma F_y = 0; \qquad V - 100(1) = 0 \qquad V = 100\text{ kN}$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad 100(1)(0.5) - M = 0 \quad M = 50\text{ kN}\cdot\text{m}$$

The normal stress developed is contributed by bending stress only. For point A, $y = 0.155 - 0.075 = 0.08$ m. Thus

$$\sigma = \frac{My}{I} = \frac{50(10^3)\,(0.08)}{37.6667(10^{-6})} = 106\text{ MPa}$$

Example



The shear stress is contributed by the transverse shear stress only. Thus,

$$\tau = \frac{VQ_A}{It} = \frac{100(10^3)[0.17625(10^{-3})]}{37.6667(10^{-6})\ (0.02)} = 23.40(10^6)\text{Pa} = 23.40\ \text{MPa}$$

The state of stress of point A can be represented by the element shown in Fig. c.

Here, $\sigma_x = -106.19$ MPa, $\sigma_y = 0$ and $\tau_{xy} = 23.40$ MPa.

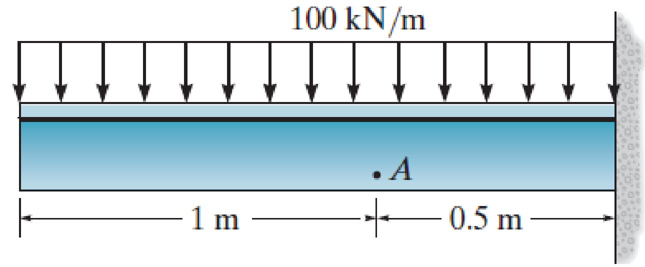
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$= \frac{-106.19 + 0}{2} \pm \sqrt{\left(\frac{-106.19 - 0}{2}\right)^2 + 23.40^2}$$

$$= -53.10 \pm 58.02$$

$\sigma_1 = 4.93$ MPa $\qquad \sigma_2 = -111$ MPa **Ans.**

Example



$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{23.40}{(-106.19 - 0)/2} = -0.4406$$

$$\theta_p = -11.89^\circ \qquad \text{ans} \quad 78.11^\circ$$

Substitute $\theta = -11.89^\circ$,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

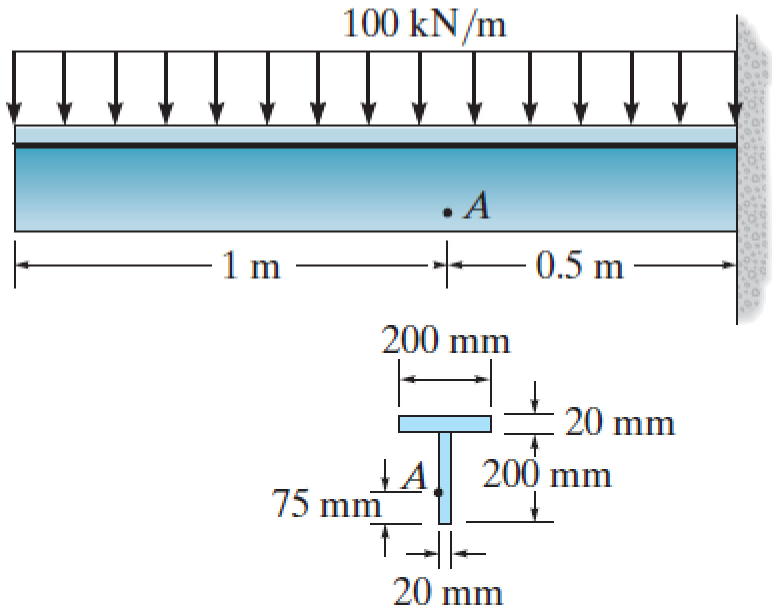
$$= \frac{-106.19 + 0}{2} + \frac{-106.19 - 0}{2}\cos(-23.78^\circ) + 23.40\ \ 5\text{m}\ (-23.78^\circ)$$

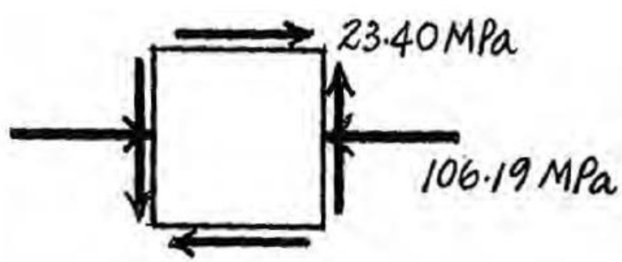
$$= -111.12\ \text{MPa} = \sigma_2$$

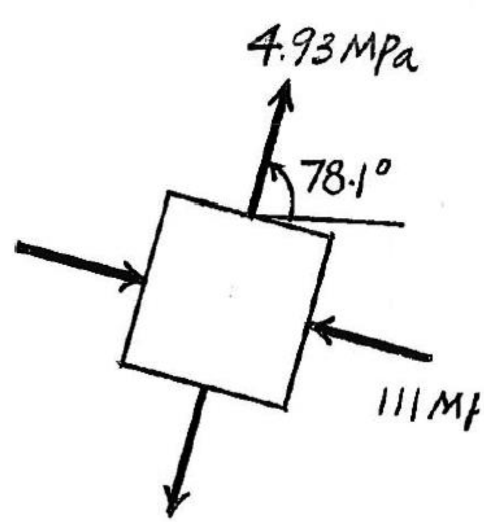
Thus,

$$(\theta_p)_1 = 78.1^\circ \qquad (\theta_p)_2 = -11.9^\circ$$

Example


100 kN/m
A
1 m
0.5 m
200 mm
20 mm
200 mm
75 mm
A
20 mm


23.40 MPa
106.19 MPa


4.93 MPa
78.1°
111 M

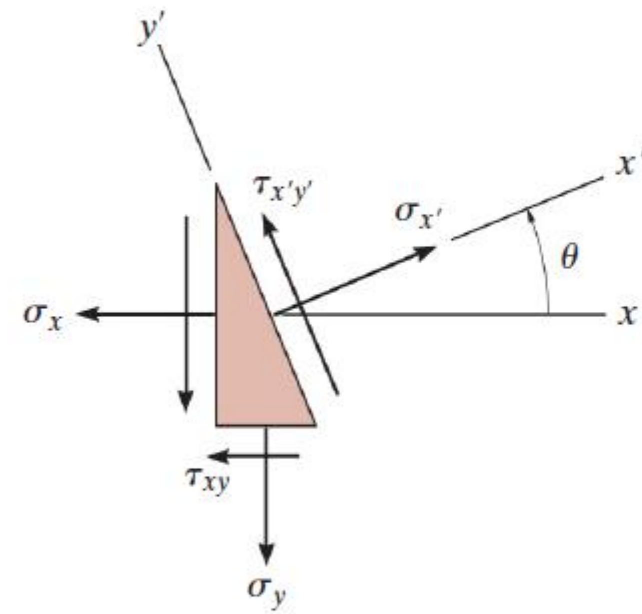
MOHR'S CIRCLE—PLANE STRESS

$$\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$(\sigma_{x'} - \sigma_{avg})^2 + \tau_{x'y'}^2 = R^2$$



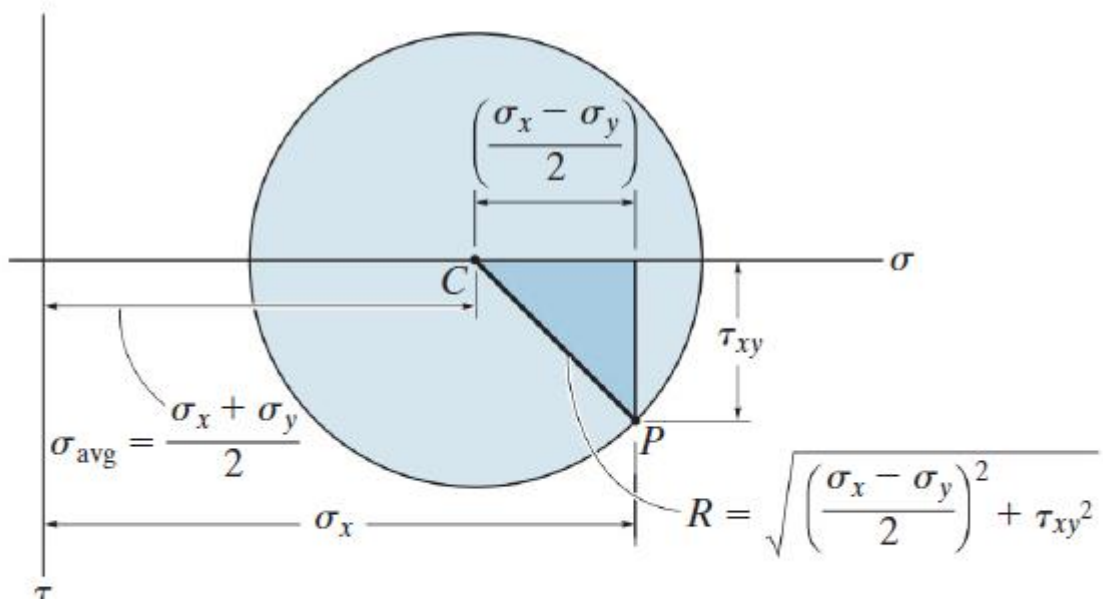
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

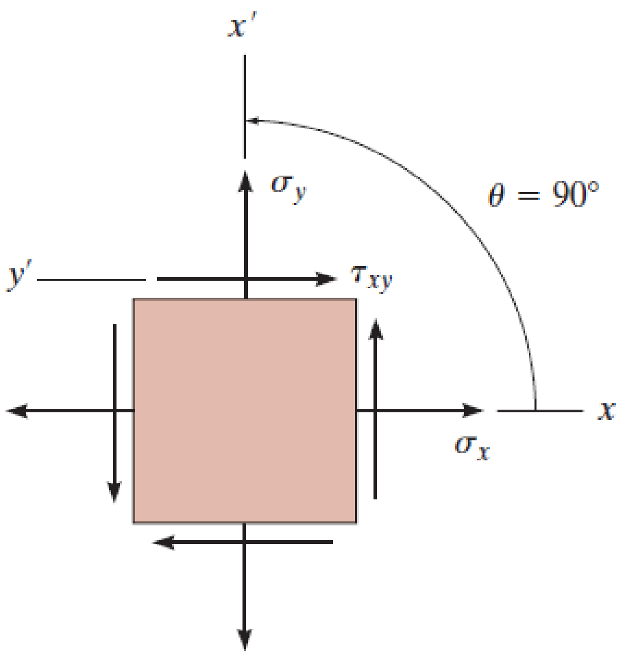
MOHR'S CIRCLE—PLANE STRESS

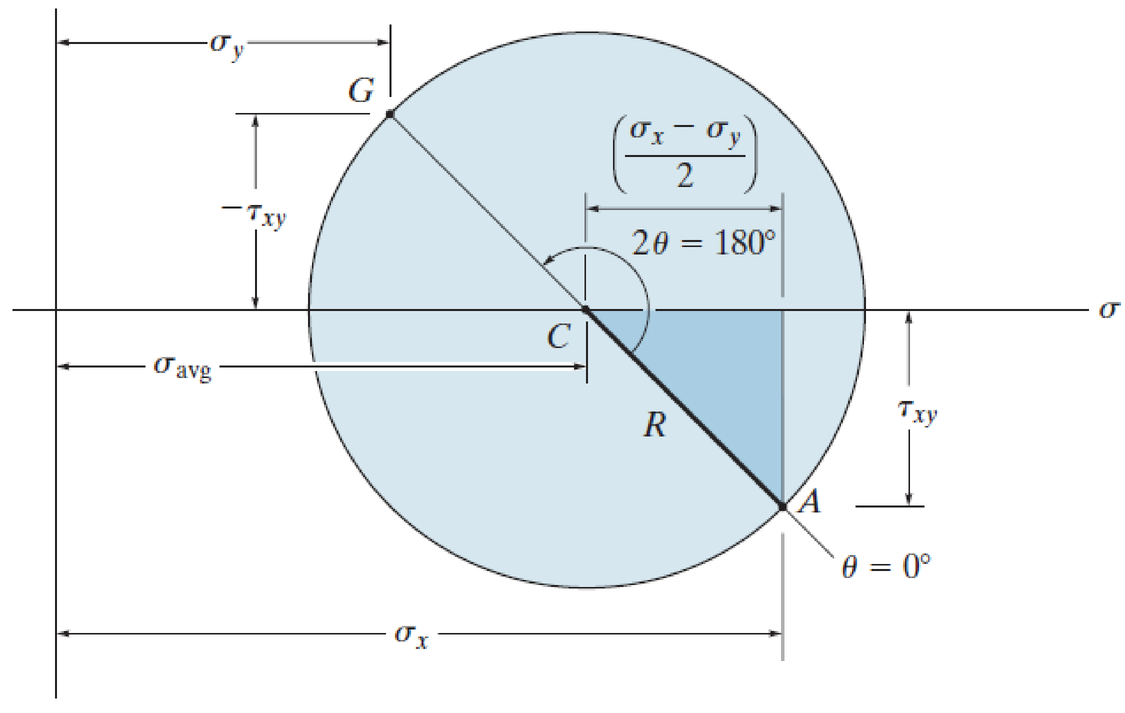
$$(\sigma_{x'} - \sigma_{avg})^2 + \tau_{x'y'}^2 = R^2$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



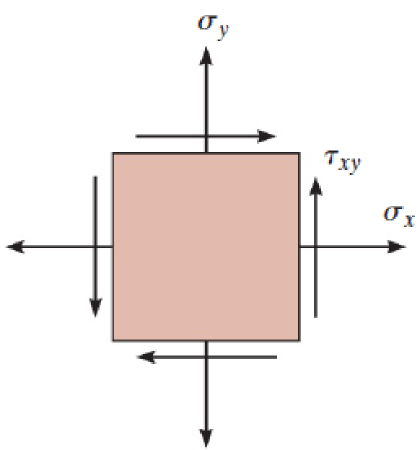
MOHR'S CIRCLE—PLANE STRESS

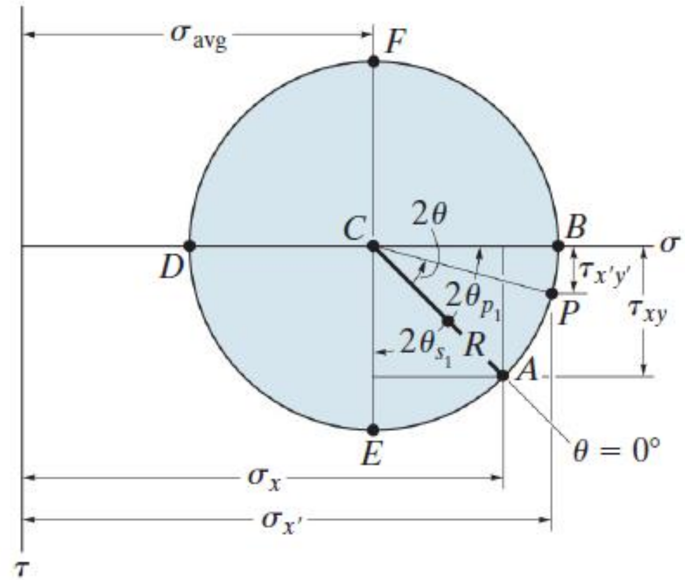


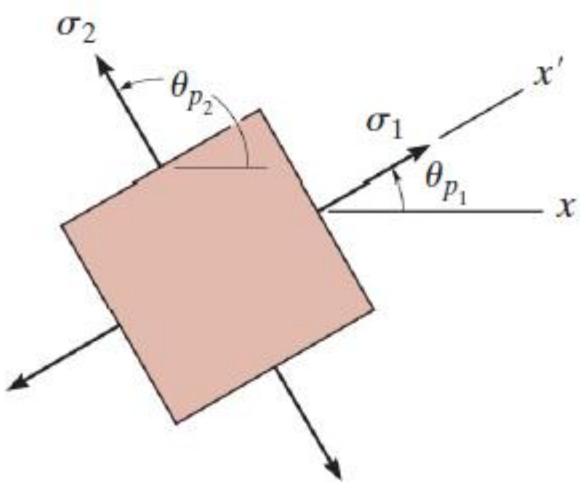


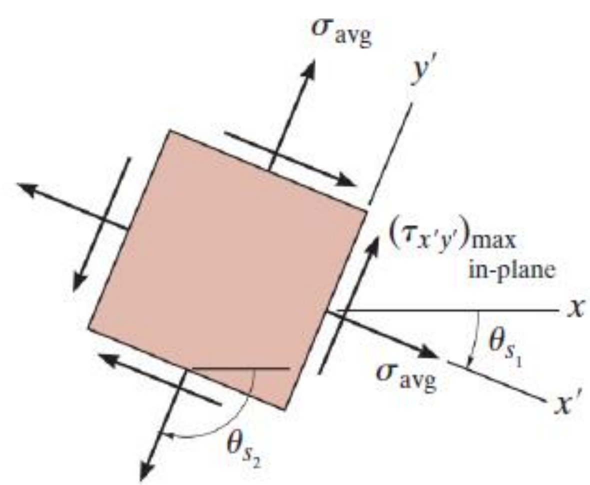
(c)

MOHR'S CIRCLE—PLANE STRESS



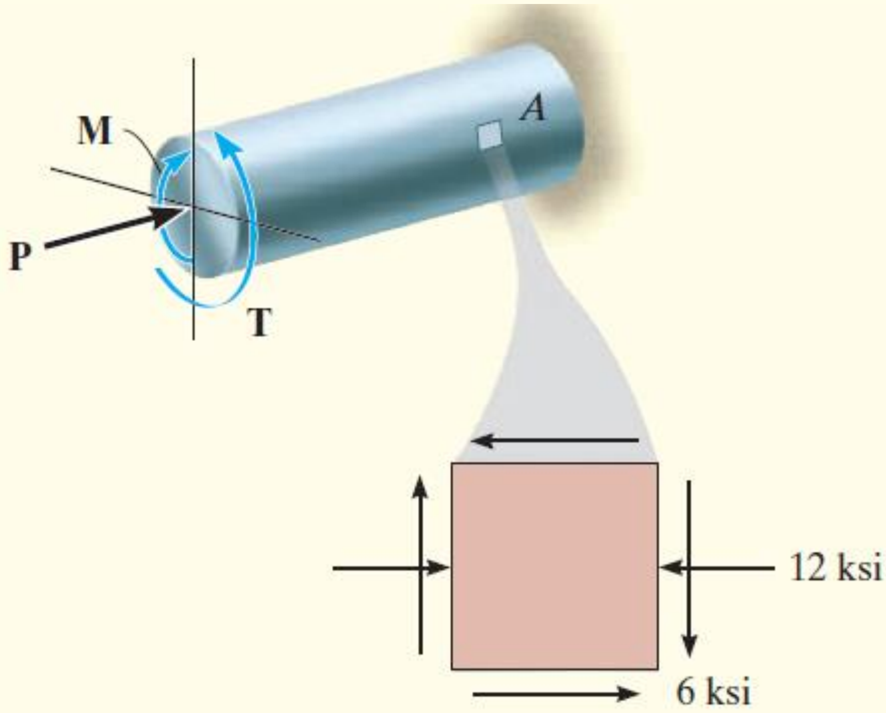




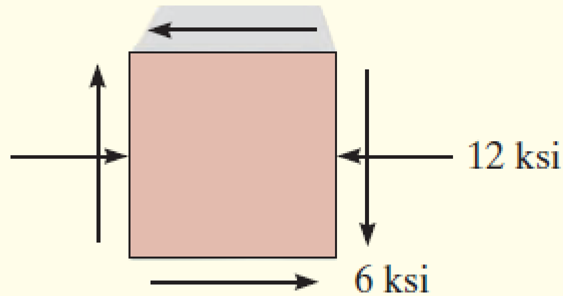


Example

Due to the applied loading, the element at point *A* on the solid shaft in Fig. 9–18*a* is subjected to the state of stress shown. Determine the principal stresses acting at this point.



Example



Construction of the Circle. From Fig. 9–18*a*,

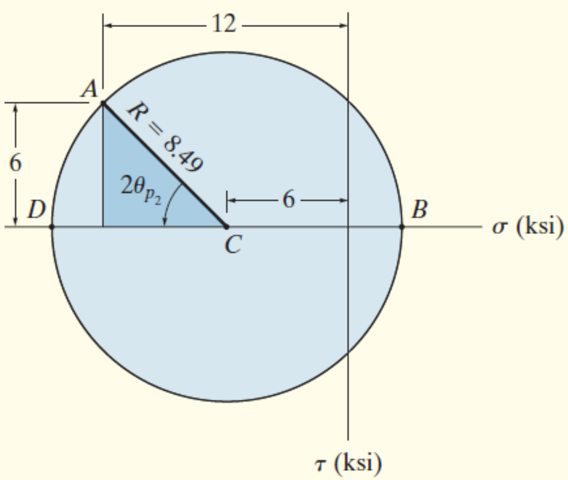
$$\sigma_x = -12 \text{ ksi} \qquad \sigma_y = 0 \qquad \tau_{xy} = -6 \text{ ksi}$$

The center of the circle is located on the σ axis at the point

$$\sigma_{avg} = \frac{-12 + 0}{2} = -6 \text{ ksi}$$

The reference point $A(-12, -6)$ and the center $C(-6, 0)$ are plotted in Fig. 9–18*b*. From the shaded triangle, the circle is constructed having a radius of

$$R = \sqrt{(12 - 6)^2 + (6)^2} = 8.49 \text{ ksi}$$



Principal Stress. The principal stresses are indicated by the coordinates of points *B* and *D*. We have, for $\sigma_1 > \sigma_2$,

$$\sigma_1 = 8.49 - 6 = 2.49 \text{ ksi} \qquad \textit{Ans.}$$

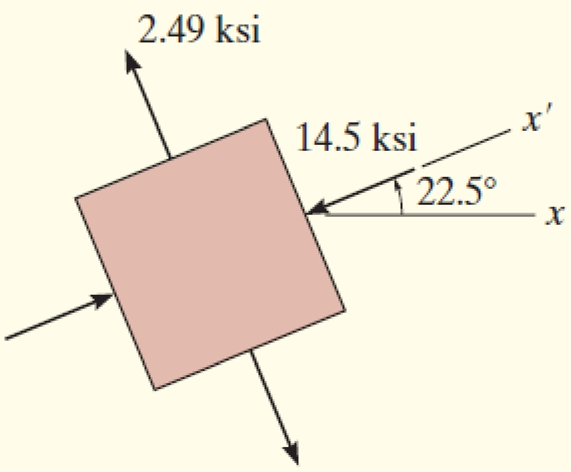
$$\sigma_2 = -6 - 8.49 = -14.5 \text{ ksi} \qquad \textit{Ans.}$$

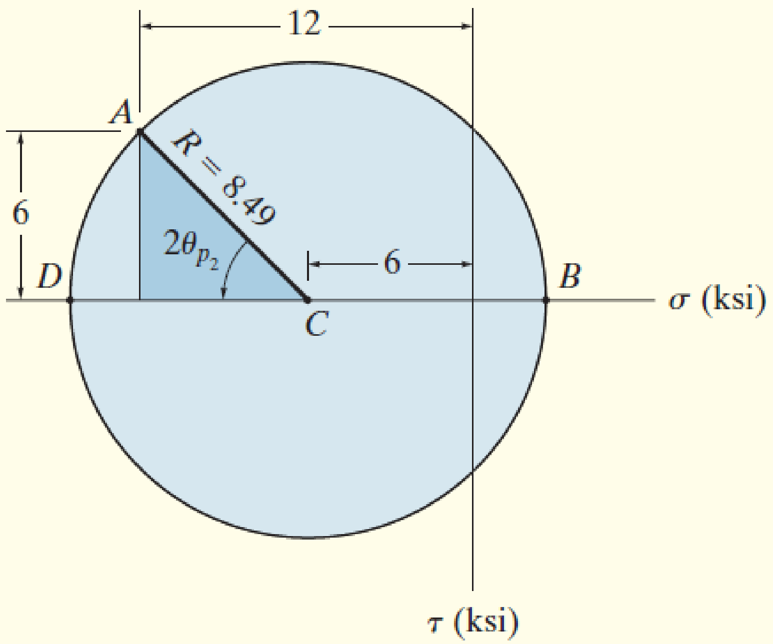
The orientation of the element can be determined by calculating the angle $2\theta_{p_2}$ in Fig. 9–18*b*, which here is measured *counterclockwise* from *CA* to *CD*. It defines the direction θ_{p_2} of σ_2 and its associated principal plane. We have

$$2\theta_{p_2} = \tan^{-1}\frac{6}{12 - 6} = 45.0^\circ$$

$$\theta_{p_2} = 22.5^\circ$$

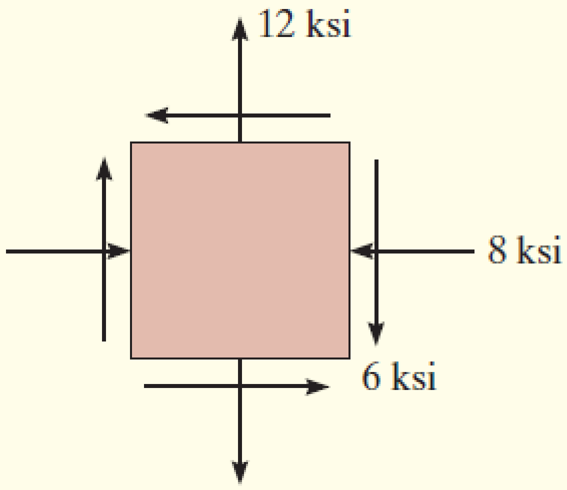
The element is oriented such that the x' axis or σ_2 is directed 22.5° *counterclockwise* from the horizontal (x axis), as shown in Fig. 9–18*c*.



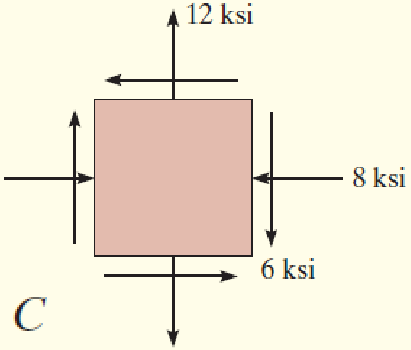


Example

The state of plane stress at a point is shown on the element in Fig. 9–20*a*. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.



Example



Construction of the Circle. From the problem data,

$$\sigma_x = -8 \text{ ksi} \qquad \sigma_y = 12 \text{ ksi} \qquad \tau_{xy} = -6 \text{ ksi}$$

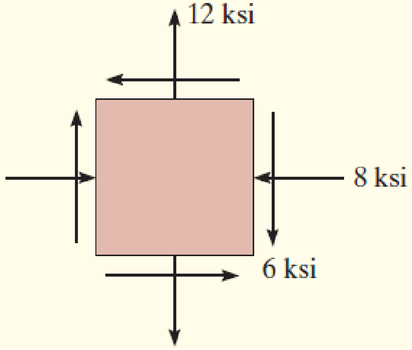
The σ and τ axes are established in Fig. 9–20*b*. The center of the circle C is on the σ axis at the point

$$\sigma_{avg} = \frac{-8 + 12}{2} = 2 \text{ ksi}$$

The reference point for $\theta = 0°$ has coordinates $A(-8, -6)$. Hence from the shaded triangle the radius CA is

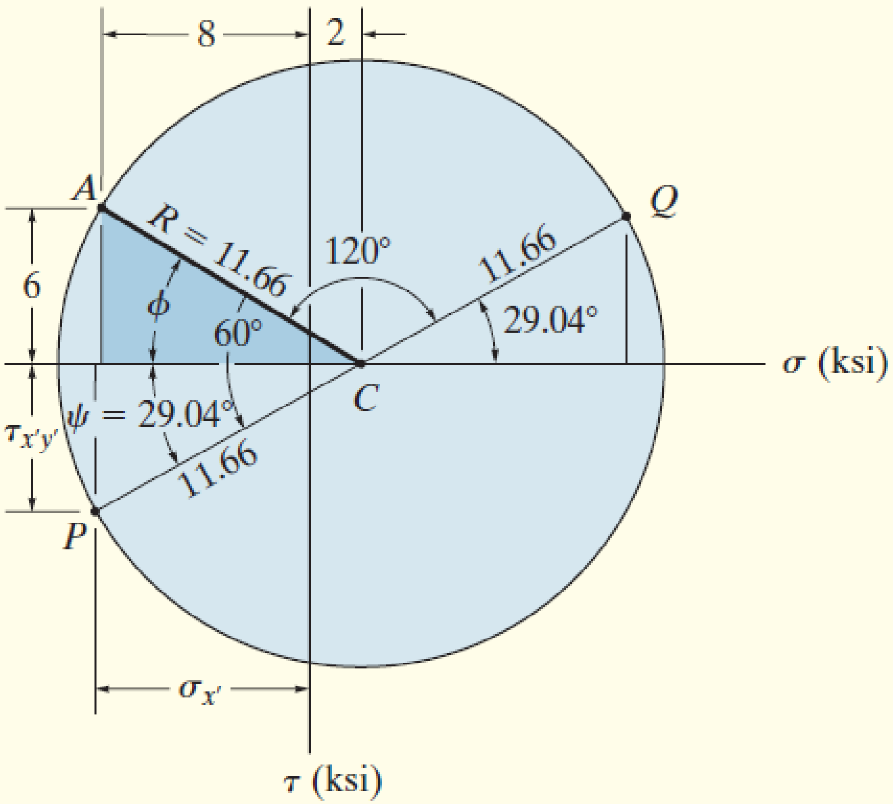
$$R = \sqrt{(10)^2 + (6)^2} = 11.66$$

Example



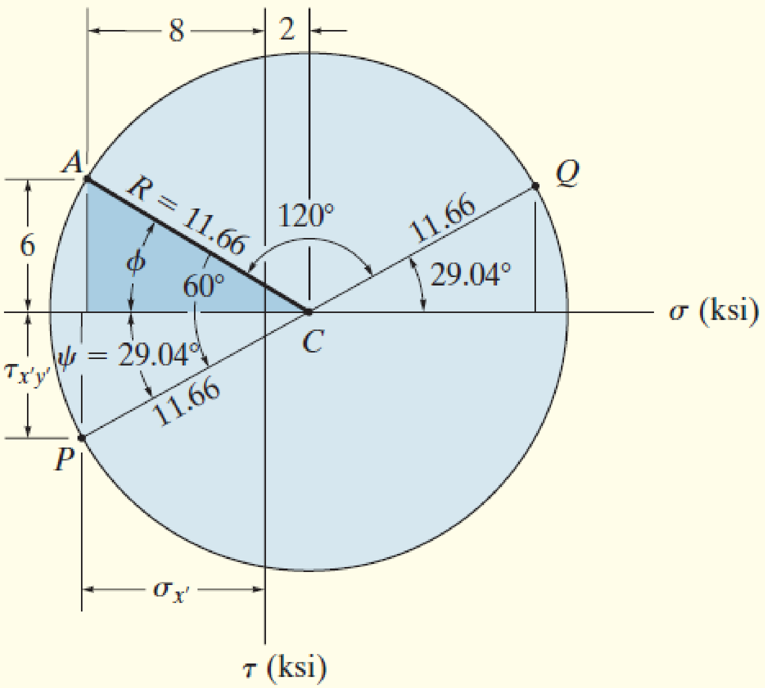
The reference point for $\theta = 0°$ has coordinates $A(-8, -6)$. Hence from the shaded triangle the radius CA is

$$R = \sqrt{(10)^2 + (6)^2} = 11.66$$



Example

Stresses on 30° Element. Since the element is to be rotated 30° *counterclockwise*, we must construct a radial line CP, $2(30°) = 60°$ *counterclockwise*, measured from CA ($\theta = 0°$), Fig. 9–20*b*. The coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$ must then be obtained. From the geometry of the circle,



$$\phi = \tan^{-1}\frac{6}{10} = 30.96°$$

$$\psi = 60° - 30.96° = 29.04°$$

$$\sigma_{x'} = 2 - 11.66\cos 29.04° = -8.20 \text{ ksi}$$

$$\tau_{x'y'} = 11.66\sin 29.04° = 5.66 \text{ ksi}$$

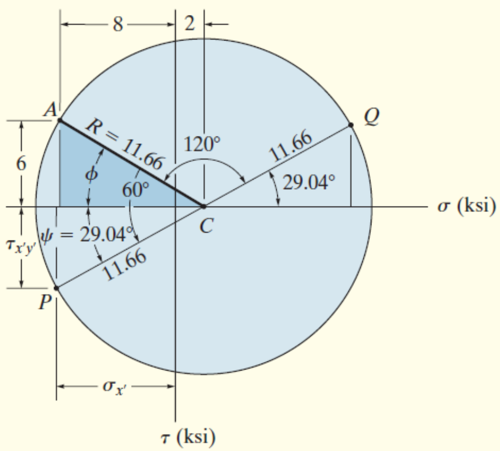
These two stress components act on face BD of the element shown in Fig. 9–20c, since the x' axis for this face is oriented 30° *counterclockwise* from the x axis.

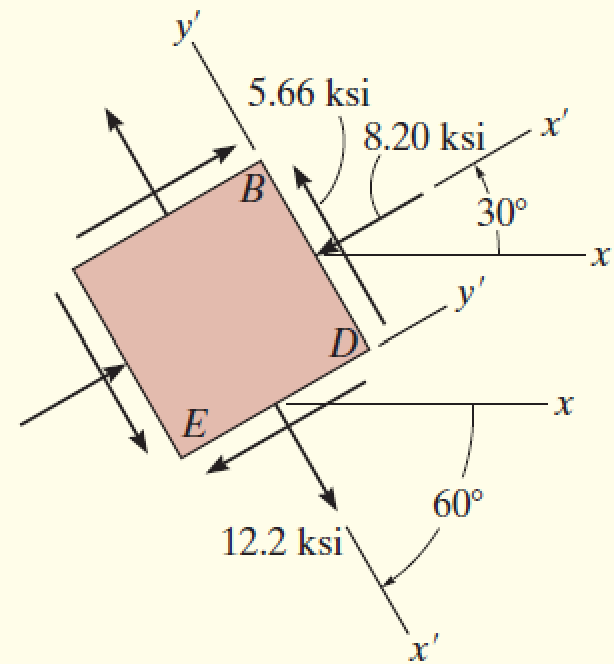
The stress components acting on the adjacent face DE of the element, which is 60° *clockwise* from the positive x axis, Fig. 9–20c, are represented by the coordinates of point Q on the circle. This point lies on the radial line CQ, which is 180° from CP, or 120° *clockwise* from CA. The coordinates of point Q are

$$\sigma_{x'} = 2 + 11.66 \cos 29.04° = 12.2 \text{ ksi}$$

$$\tau_{x'y'} = -(11.66 \sin 29.04) = -5.66 \text{ ksi} \quad \text{(check)}$$

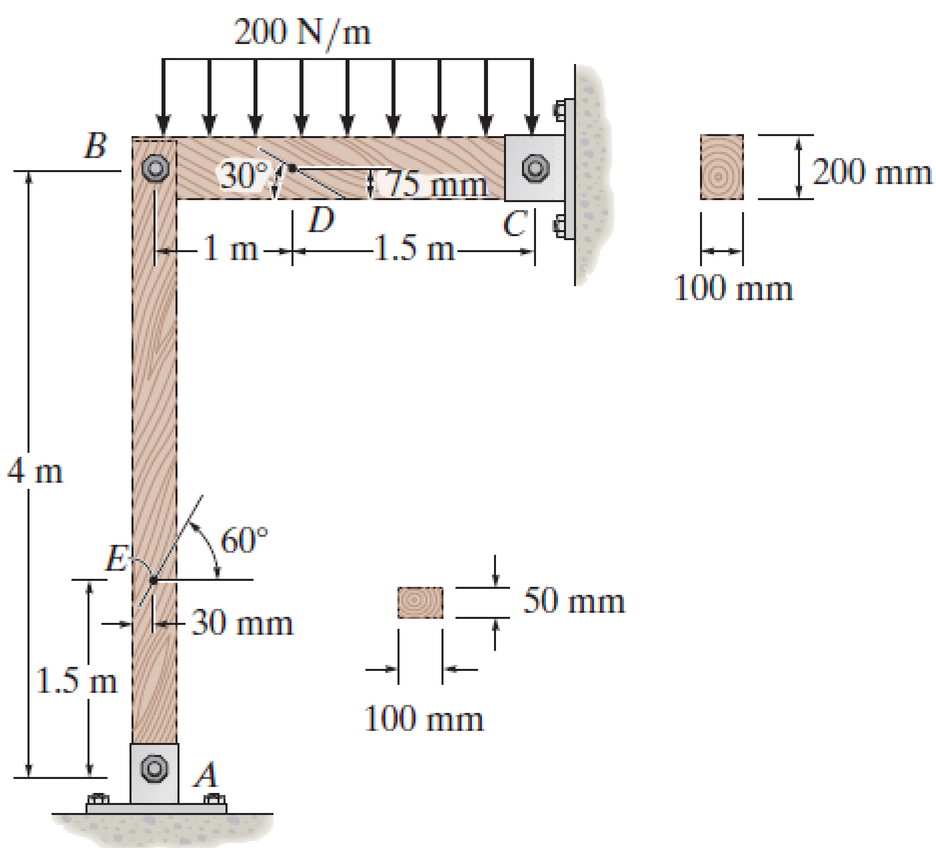
NOTE: Here $\tau_{x'y'}$ acts in the $-y'$ direction, Fig. 9–20c.





Quiz

The frame supports the triangular distributed load shown. Determine the normal and shear stresses at point *E* that act perpendicular and parallel, respectively, to the grains. The grains at this point make an angle of 45° with the horizontal as shown.

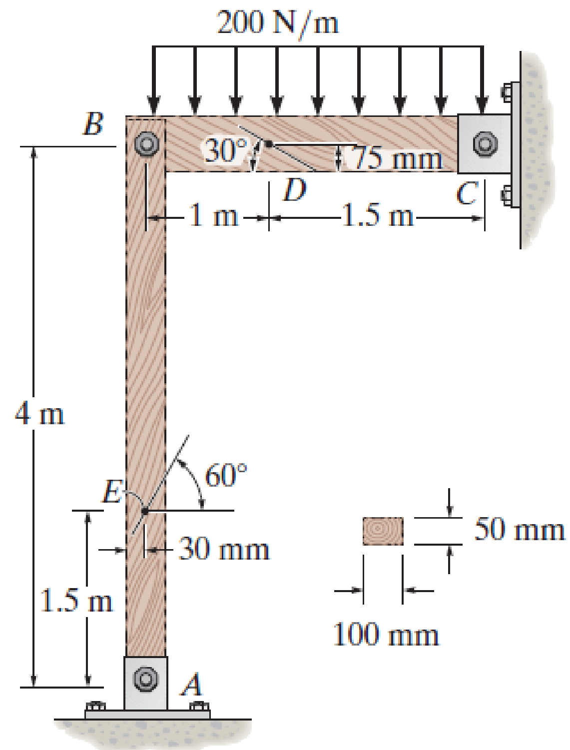


Section Properties:

$$A = 0.1(0.05) = 5.00\left(10^{-3}\right)\ \mathrm{m}^2$$

Normal Stress:

$$\sigma_E = \frac{N}{A} = \frac{-250}{5.00(10^{-3})} = -50.0\ \mathrm{kPa}$$



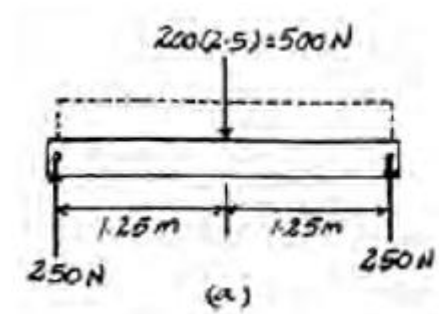
Construction of the Circle: In accordance with the sign convention. $\sigma_x = 0$, $\sigma_y = -50.0$ kPa, and $\tau_{xy} = 0$. Hence.

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{0 + (-50.0)}{2} = -25.0\ \mathrm{kPa}$$

The coordinates for reference points A and C are

$$A(0, 0) \qquad C(-25.0, 0)$$

The radius of circle is $R = 25.0 - 0 = 25.0$ kPa



Stress on the Rotated Element:

$$\sigma_x = -25.0 + 25.0 \cos 60^\circ = -12.5 \text{ kPa}$$

$$\tau_{x'y'} = 25.0 \sin 60^\circ = 21.7 \text{ kPa}$$