Chapter 7:
Centroids and Centers of Gravity

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Introduction

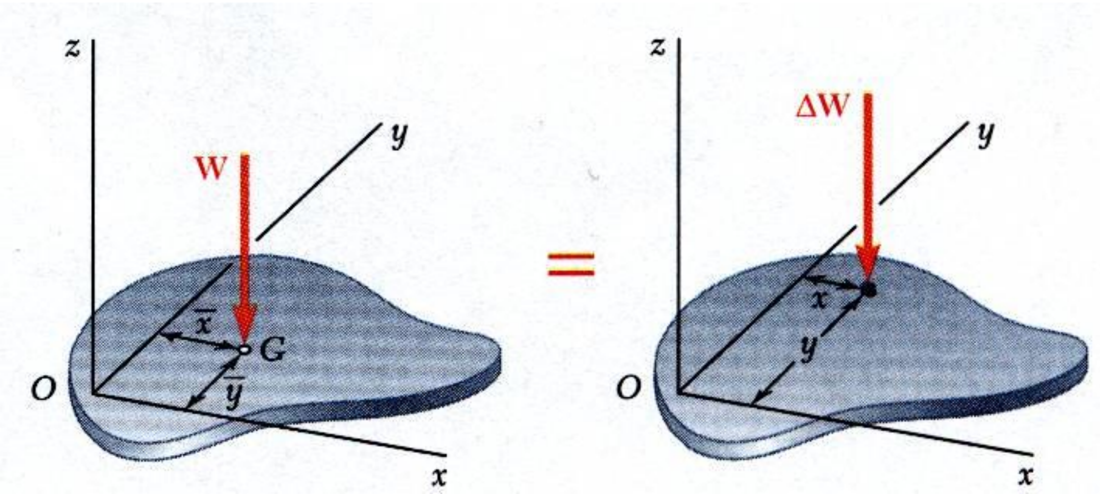
- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the ***center of gravity*** for the body.

- The ***centroid*** *of an area* is analogous to the center of gravity of a body. The concept of the ***first moment of an area*** is used to locate the centroid.

- Determination of the area of a ***surface of revolution*** and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

Center of Gravity of a 2D Body

- Center of gravity of a plate



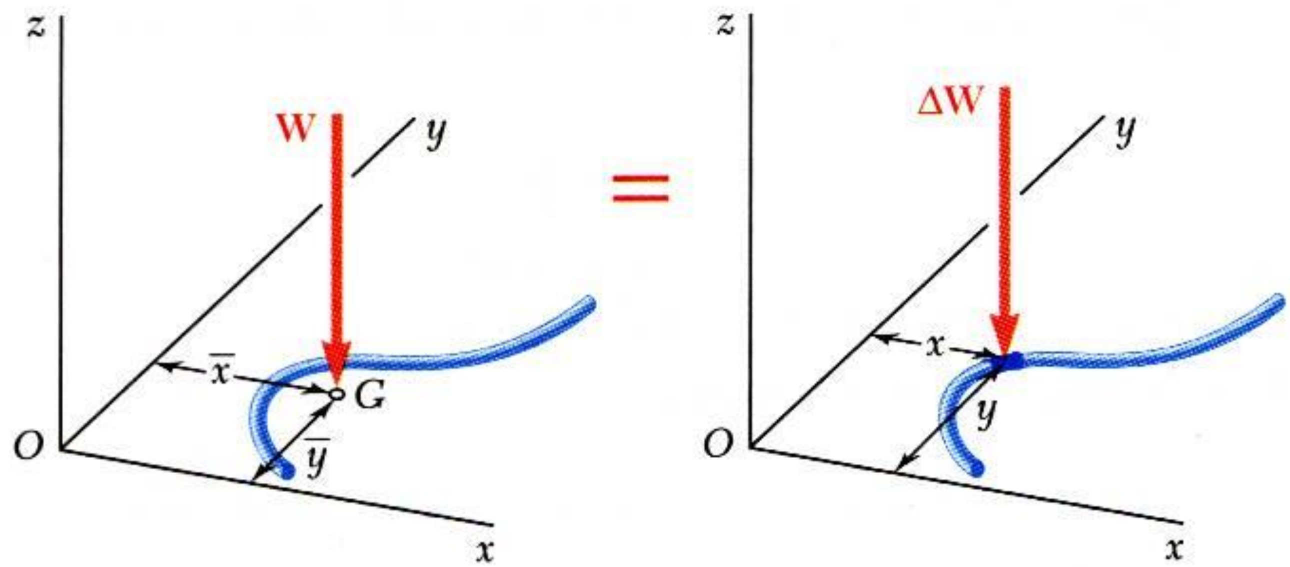
$$\sum F_z : \quad \Rightarrow \quad W = \Delta W_1 + \Delta W_1 + \cdots + \Delta W_n$$

$$\sum M_y : \quad \Rightarrow \quad \bar{x}W = \sum x\Delta W \quad \Rightarrow \quad \bar{x} = \frac{\sum x\Delta W}{W} \quad or \quad \bar{x} = \frac{\int x\,dW}{W}$$

$$\sum M_x : \quad \Rightarrow \quad \bar{y}W = \sum y\Delta W \quad \Rightarrow \quad \bar{y} = \frac{\sum y\Delta W}{W} \quad or \quad \bar{y} = \frac{\int y\,dW}{W}$$

Center of Gravity of a 2D Body

- Center of gravity of a wire

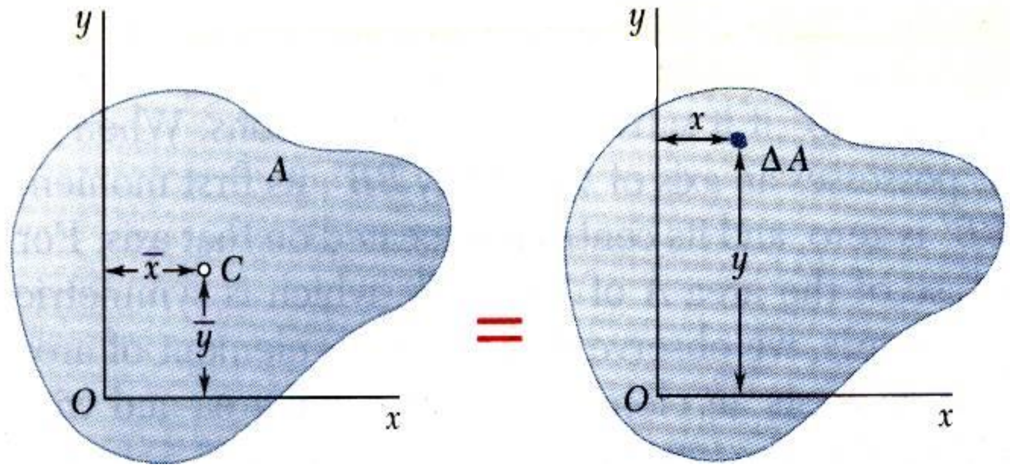


$$\sum M_y: \Rightarrow \bar{x}W = \sum x\Delta W \Rightarrow \quad \bar{x} = \frac{\sum x\Delta W}{W} \quad or \quad \bar{x} = \frac{\int x\,dW}{W}$$

$$\sum M_x: \Rightarrow \bar{y}W = \sum y\Delta W \Rightarrow \quad \bar{y} = \frac{\sum y\Delta W}{W} \quad or \quad \bar{y} = \frac{\int y\,dW}{W}$$

Centroids and First Moments of Areas

- Centroid of an area



$$\Delta W = \gamma\, t\, \Delta A \quad \Rightarrow \quad \boxed{W = \gamma\, t\, A}$$

$$\bar{x} = \frac{\int x\, dW}{W} \quad \Rightarrow \quad \bar{x} = \frac{\int x(\gamma t) dA}{(\gamma A t)} \quad \Rightarrow \quad \boxed{\bar{x} = \frac{\int x\, dA}{A}}$$

$$\text{if} \quad \boxed{Q_y = \int x\, dA \quad \Rightarrow \quad Q_y = \bar{x} A}$$

Q_y : First moment with respect to y axis

$$\bar{y} = \frac{\int y\, dW}{W} \quad \Rightarrow \quad \bar{y} = \frac{\int y(\gamma t) dA}{(\gamma A t)} \quad \Rightarrow \quad \boxed{\bar{y} = \frac{\int y\, dA}{A}}$$

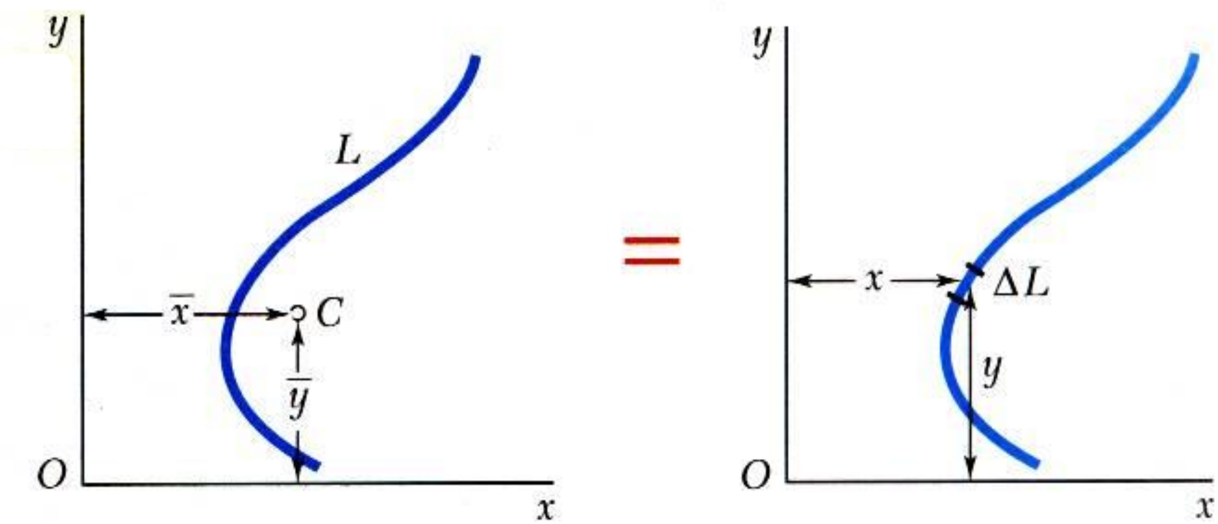
$$\text{if} \quad \boxed{Q_x = \int y\, dA \quad \Rightarrow \quad Q_x = \bar{y} A}$$

Q_x : First moment with respect to x axis

Centroids and First Moments of Lines

- Centroid of a line

$$\Delta W = \gamma\, a\, \Delta L \quad \Rightarrow \quad \boxed{W = \gamma\, a\, L}$$



$a:$ Cross section area

$$\bar{x} = \frac{\int x\, dW}{W} \quad \Rightarrow \quad \bar{x} = \frac{\int x(\gamma a)\, dL}{(\gamma L a)} \quad \Rightarrow \quad \boxed{\bar{x} = \frac{\int x\, dL}{L}}$$

$$\bar{y} = \frac{\int y\, dW}{W} \quad \Rightarrow \quad \bar{y} = \frac{\int y(\gamma a)\, dL}{(\gamma L a)} \quad \Rightarrow \quad \boxed{\bar{y} = \frac{\int y\, dL}{L}}$$

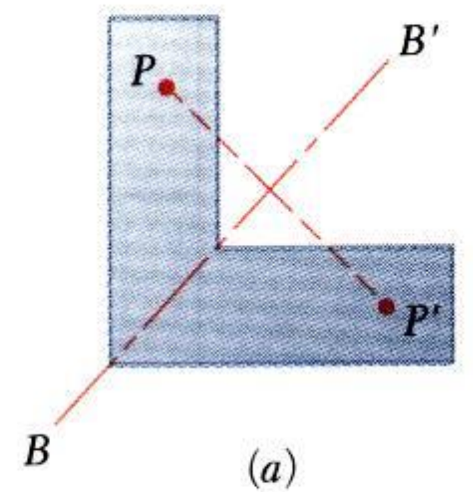
$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

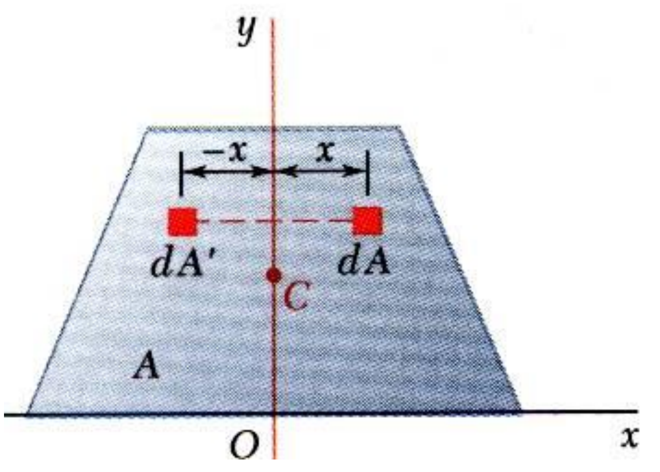
First Moments of Areas and Lines

- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.



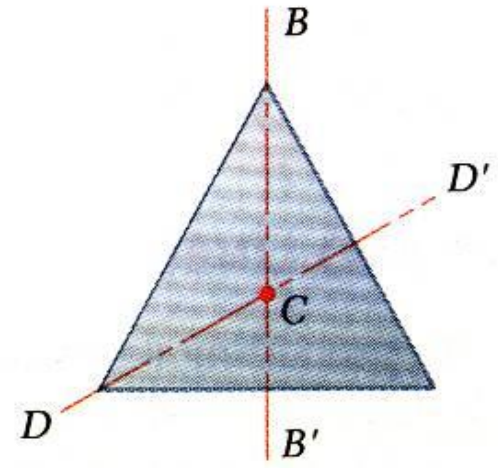
- The first moment of an area with respect to a line of symmetry is zero.

- If an area possesses a line of symmetry, its centroid lies on that axis

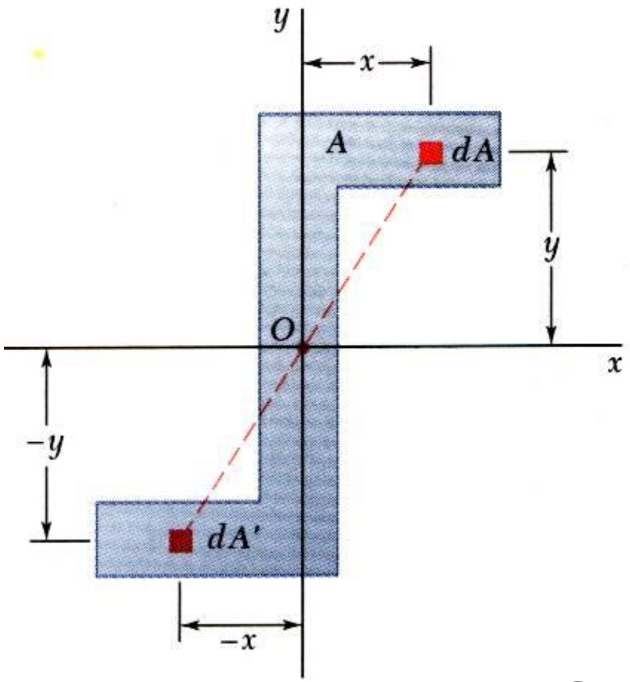


First Moments of Areas and Lines

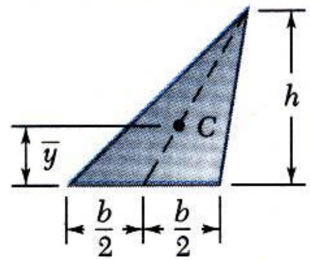
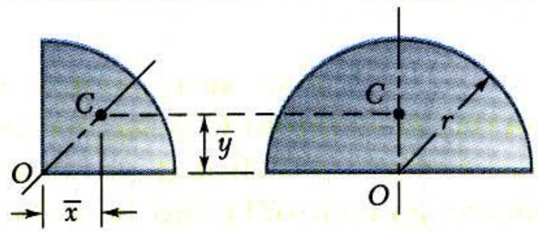
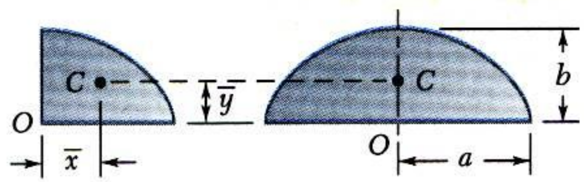
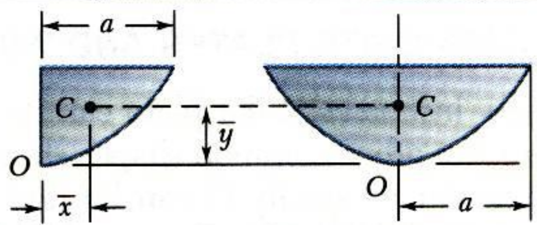
- If an area possesses two lines of symmetry, its centroid lies at their intersection.



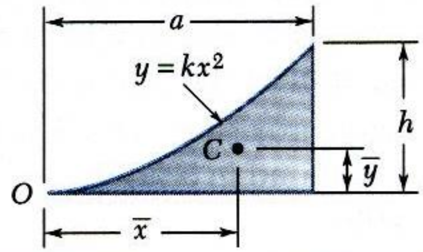
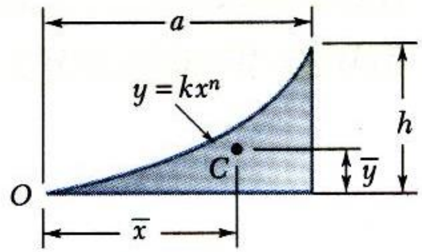
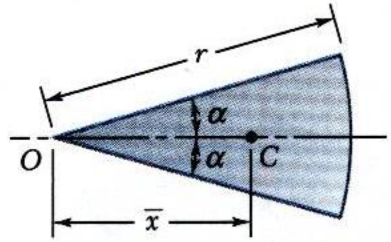
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.



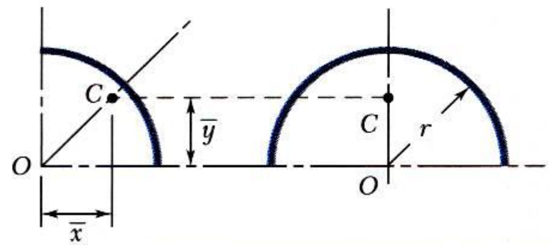
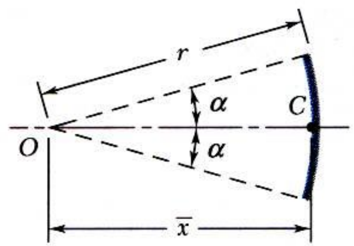
❑ Centroids of Common Shapes of Areas

Shape	Figure	$\overline{x}$	$\overline{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$

Centroids of Common Shapes of Areas

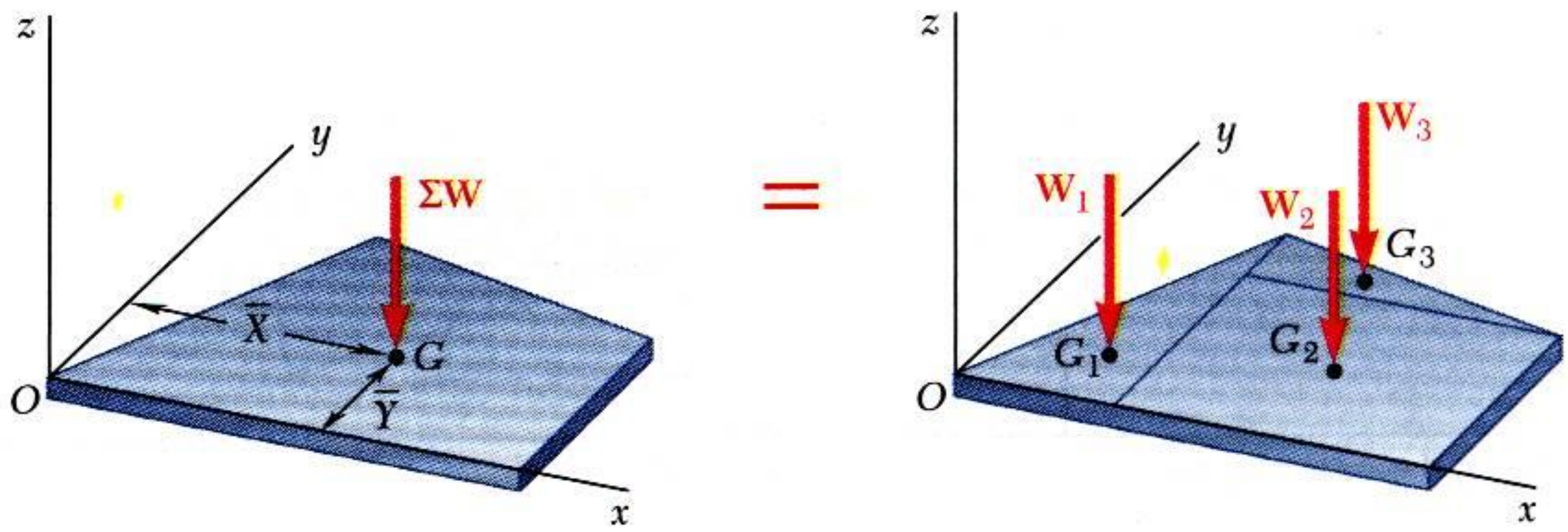
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

❑ Centroids of Common Shapes of Lines

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Composite Plates

- Composite plates

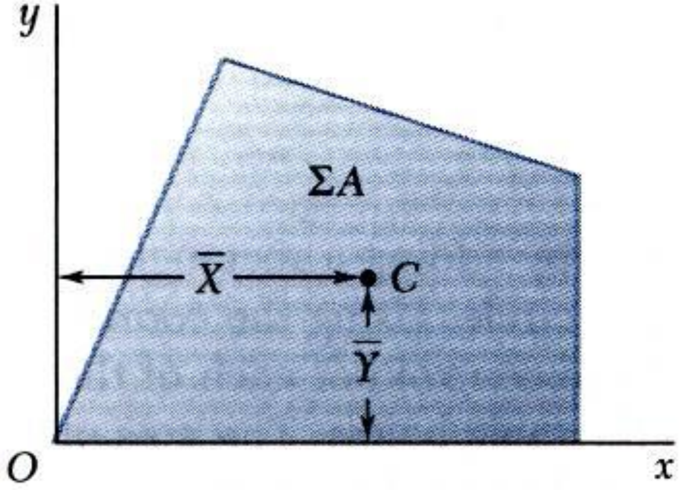


$$\bar{X} = \frac{\sum \bar{x}W}{\sum W}$$

$$\bar{Y} = \frac{\sum \bar{y}W}{\sum W}$$

Composite Areas

- Composite area



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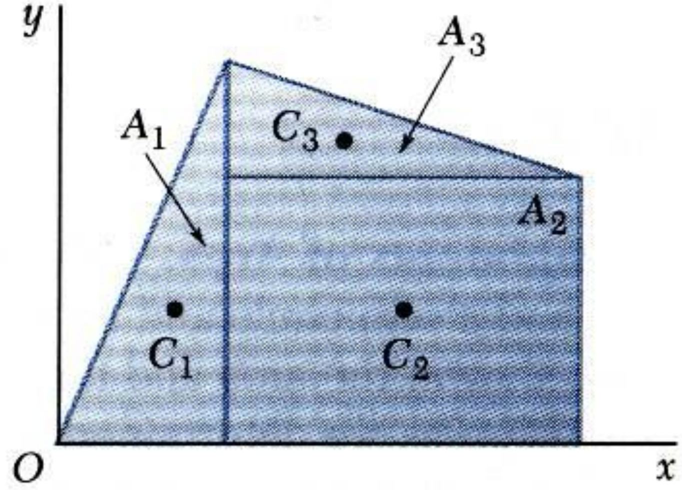
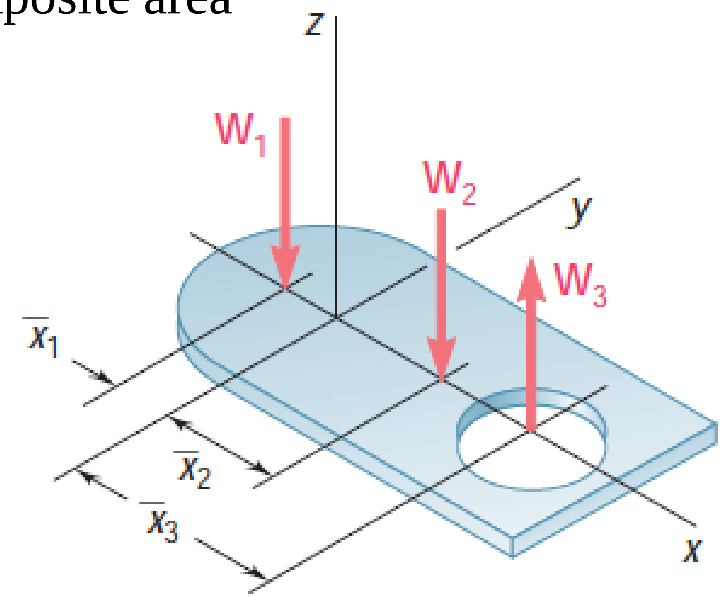


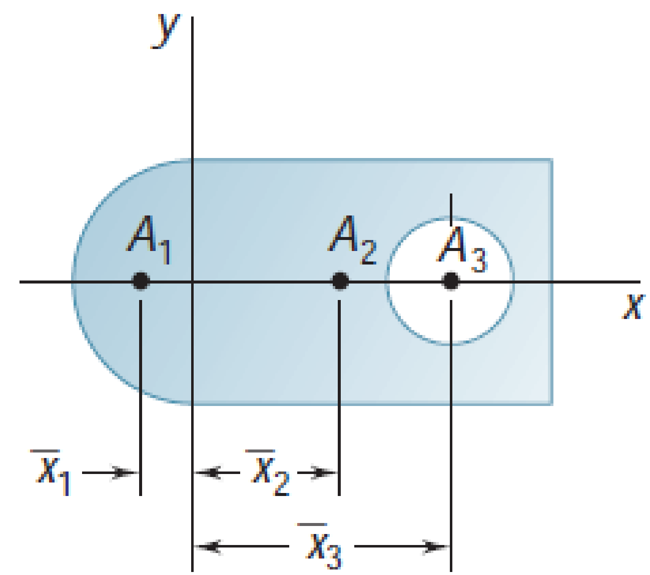
$$\bar{X} = \frac{\sum \bar{x} A}{\sum A}$$

$$\bar{Y} = \frac{\sum \bar{y} A}{\sum A}$$

❑ Composite Areas

- Composite area

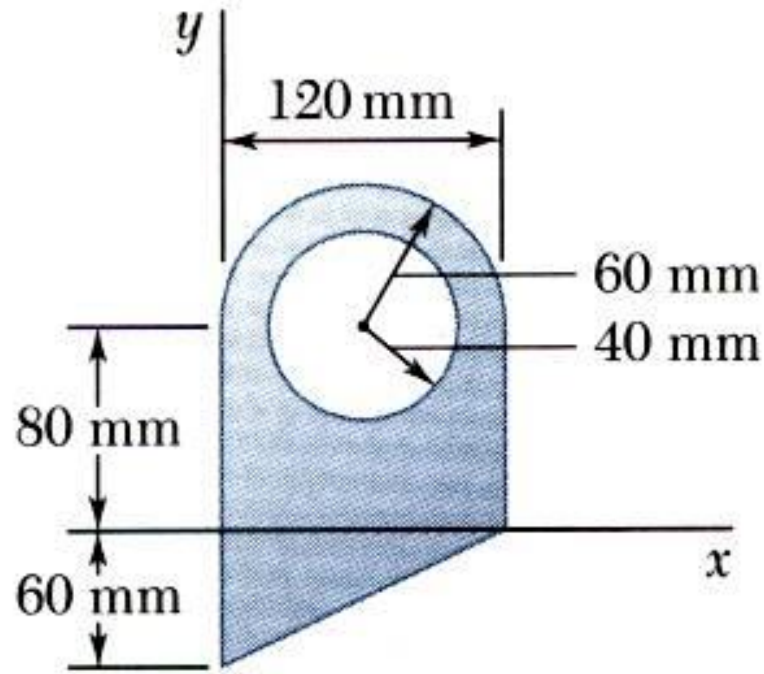




	$\bar{x}$	A	$\bar{x}A$
A_1 Semicircle	−	+	−
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	−	−

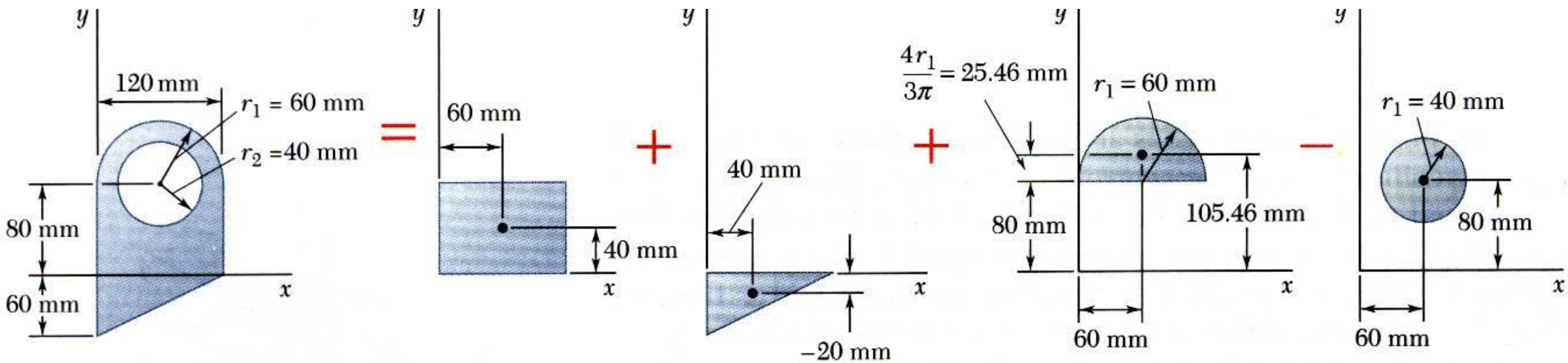
❑ Sample Problem 01

For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.



Sample Problem 01

SOLUTION:



Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2\times10^3\,\text{mm}^3$$
$$Q_y = +757.7\times10^3\,\text{mm}^3$$

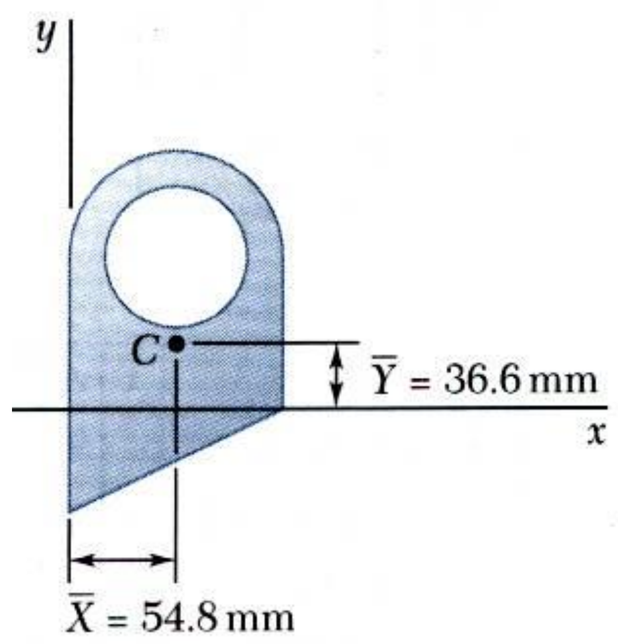
Sample Problem 01

SOLUTION:

- Compute the coordinates of the area centroid by dividing the first moments by the total area.

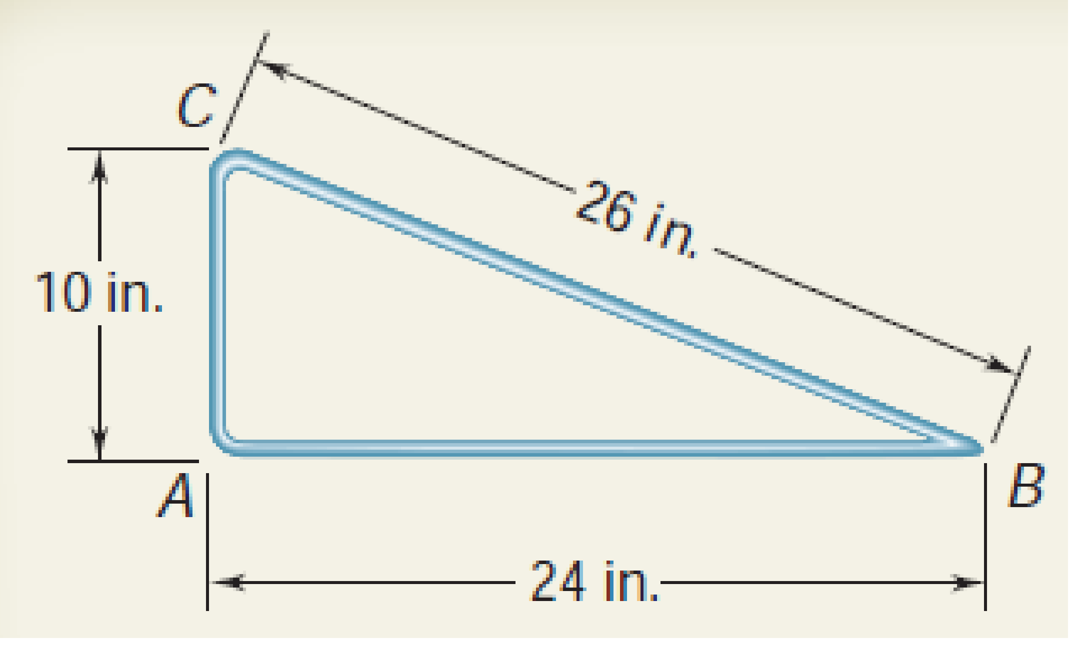
$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2} \Rightarrow \boxed{\bar{X} = 54.8\,(mm)}$$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2} \Rightarrow \boxed{\bar{Y} = 36.6\,(mm)}$$



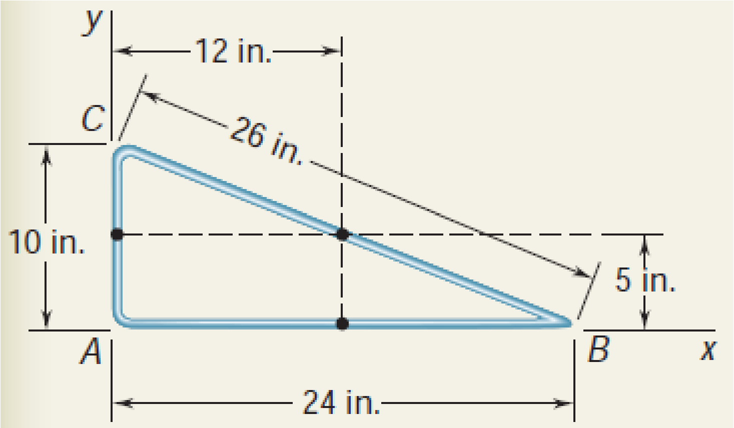
Sample Problem 02

The figure shown is made from a piece of thin, homogeneoius wire Determine the location of its center of gravity.



Sample Problem 02

SOLUTION:



Segment	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
	$\Sigma L = 60$			$\Sigma \bar{x}L = 600$	$\Sigma \bar{y}L = 180$

$$\bar{X} = \frac{\sum \bar{x}L}{\sum L} = \frac{600}{60} \Rightarrow \boxed{\bar{X} = 10\ (in.)}$$

$$\bar{Y} = \frac{\sum \bar{y}L}{\sum L} = \frac{180}{60} \Rightarrow \boxed{\bar{Y} = 3\ (in.)}$$

❑ Determination of Centroids by Integration

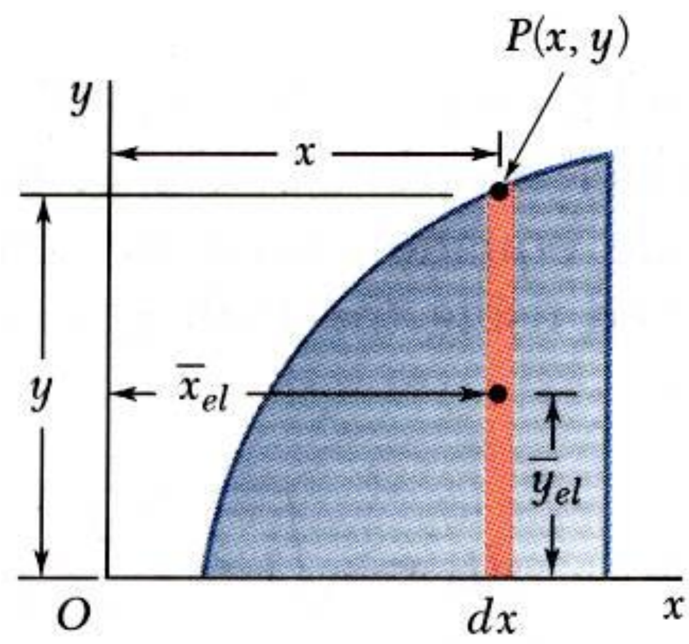
- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.

$$Q_y = \bar{x}A = \int x dA \;\Rightarrow\; \boxed{Q_y = \iint x\,dxdy \quad or \quad Q_y = \int \bar{x}_{el}\,dA}$$

$$Q_x = \bar{y}A = \int y dA \;\Rightarrow\; \boxed{Q_x = \iint y\,dxdy \quad or \quad Q_x = \int \bar{y}_{el}\,dA}$$

$$Q_y = \bar{x}A = \int \bar{x}_{el}\,dA \;\Rightarrow\; \boxed{Q_y = \int x(ydx)}$$

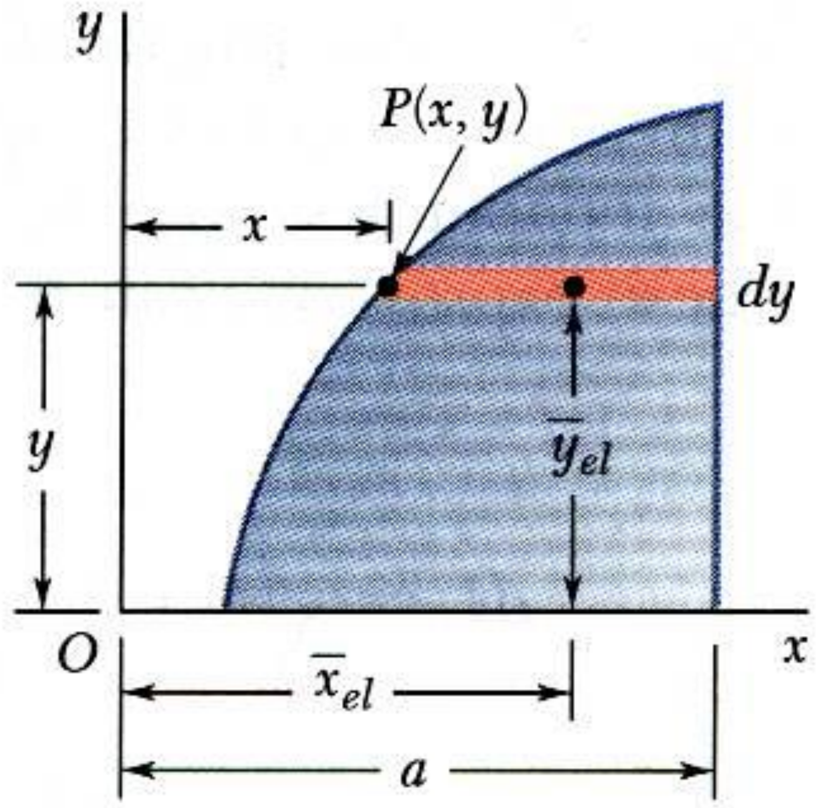
$$Q_x = \bar{y}A = \int \bar{y}_{el}\,dA \;\Rightarrow\; \boxed{Q_x = \int \frac{y}{2}(ydx)}$$



Determination of Centroids by Integration

$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA \quad \Rightarrow \quad \boxed{Q_y = \int \frac{a+x}{2}[(a-x)\, dy]}$$

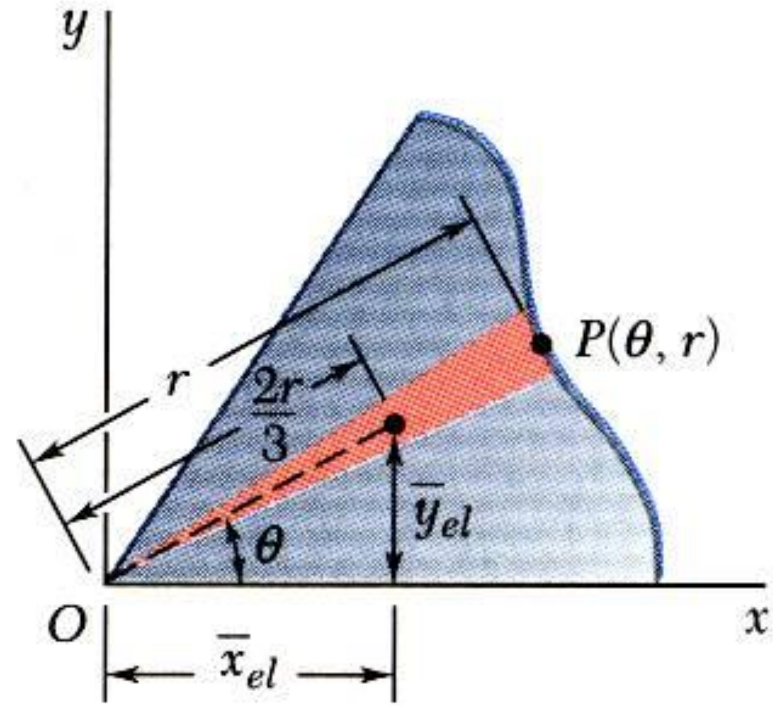
$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA \quad \Rightarrow \quad \boxed{Q_x = \int y[(a-x)\, dy]}$$



Determination of Centroids by Integration

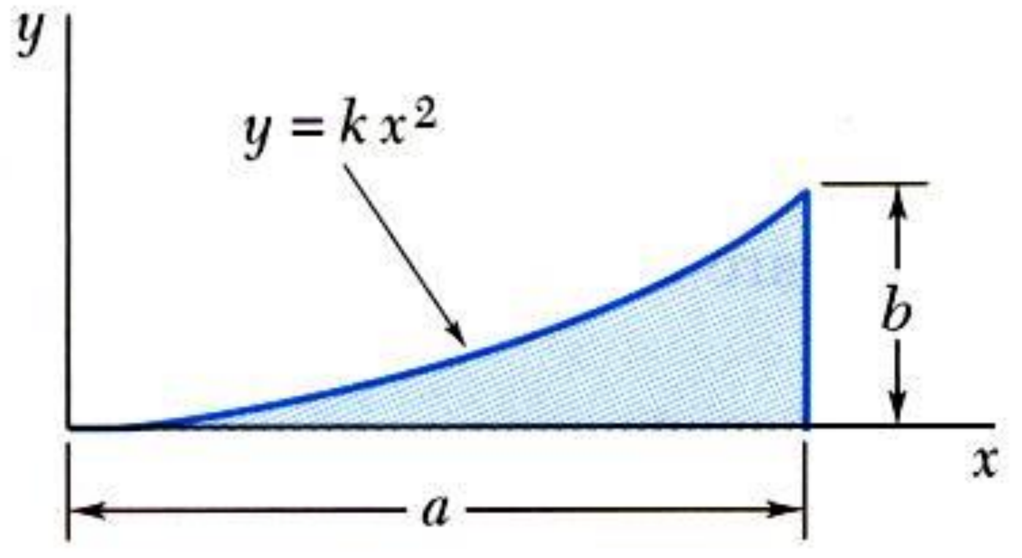
$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA \quad \Rightarrow \quad Q_y = \int \frac{2r}{3}\cos\theta \left(\frac{1}{2}r^2 d\theta\right)$$

$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA \quad \Rightarrow \quad Q_x = \int \frac{2r}{3}\sin\theta \left(\frac{1}{2}r^2 d\theta\right)$$



❑ Sample Problem 03

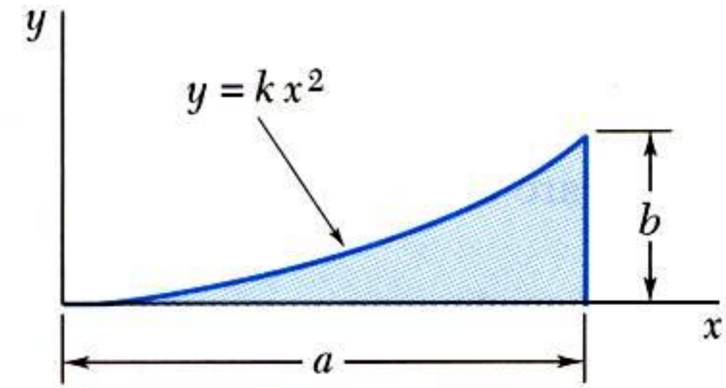
Determine by direct integration the location of the centroid of a parabolic spandrel.

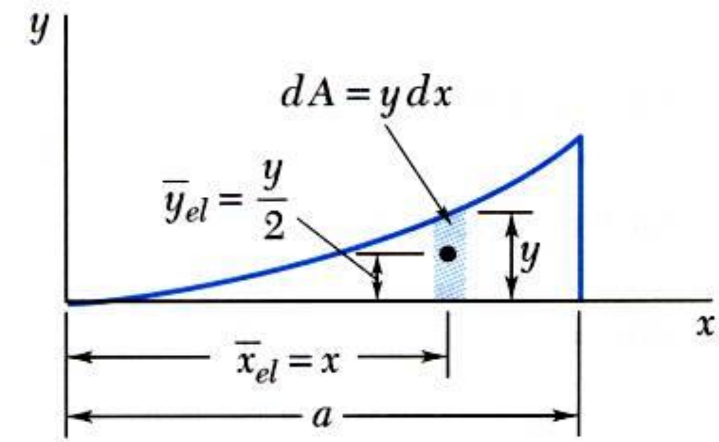


❑ Sample Problem 03

SOLUTION:

- Determine the constant k.

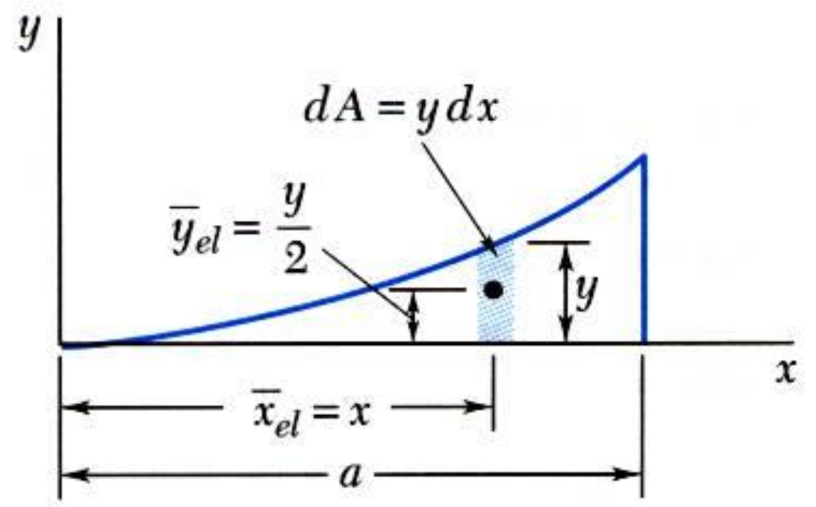




Sample Problem 03

SOLUTION:

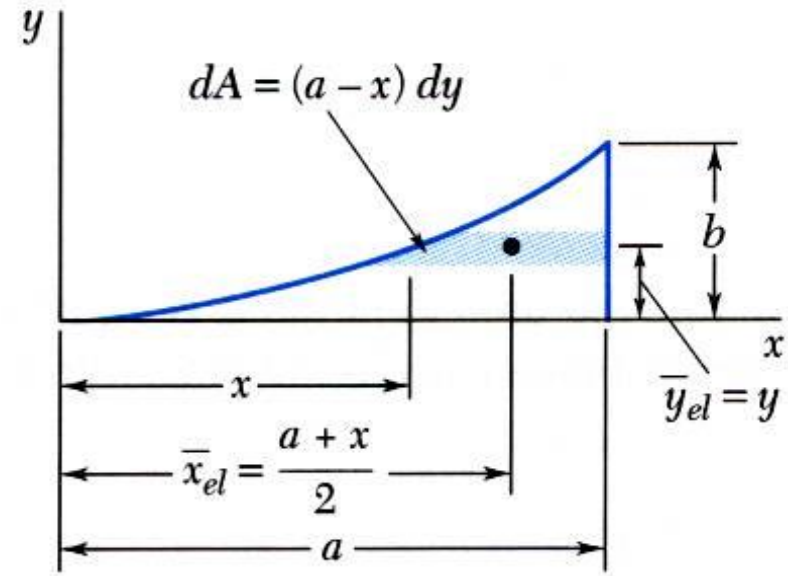
- Using vertical strips, perform a single integration to find the first moments.



Sample Problem 03

SOLUTION:

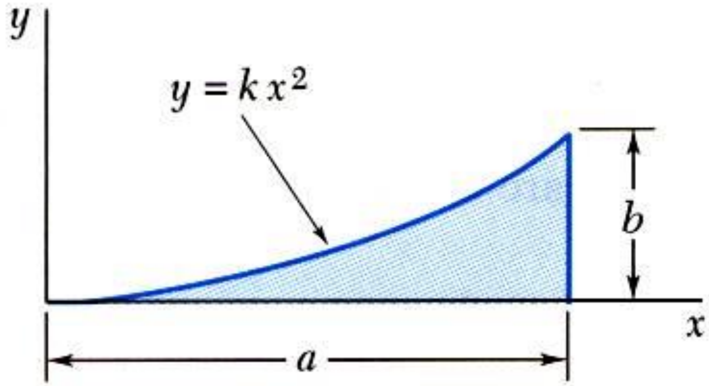
- Or, using horizontal strips, perform a single integration to find the first moments.



❑ Sample Problem 03

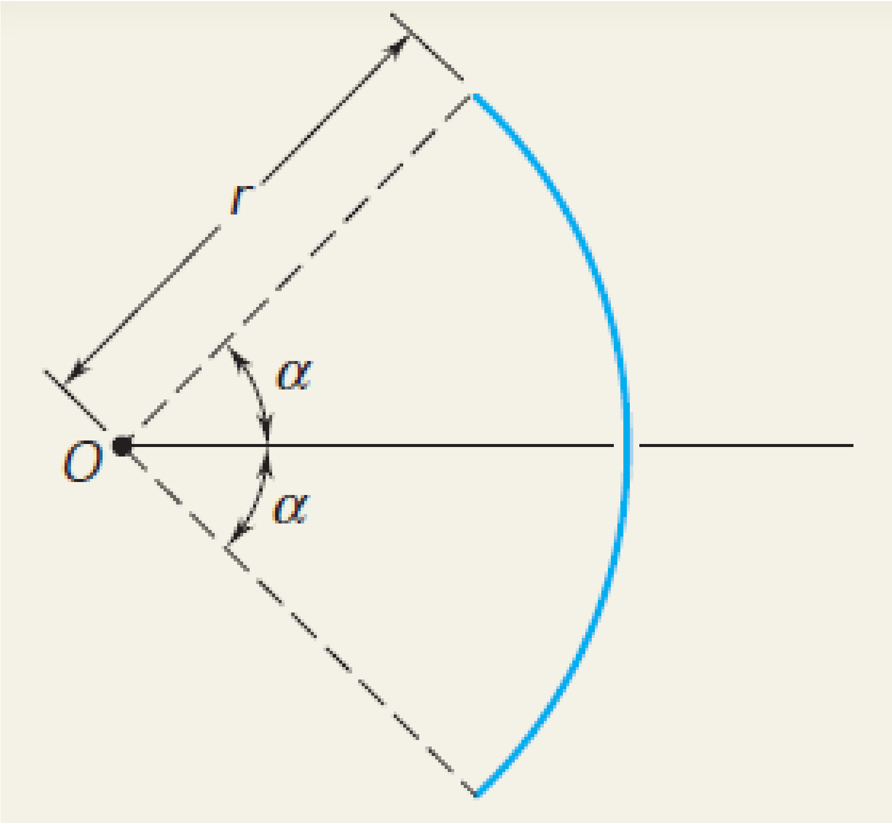
SOLUTION:

- Evaluate the centroid coordinates.



❑ Sample Problem 04

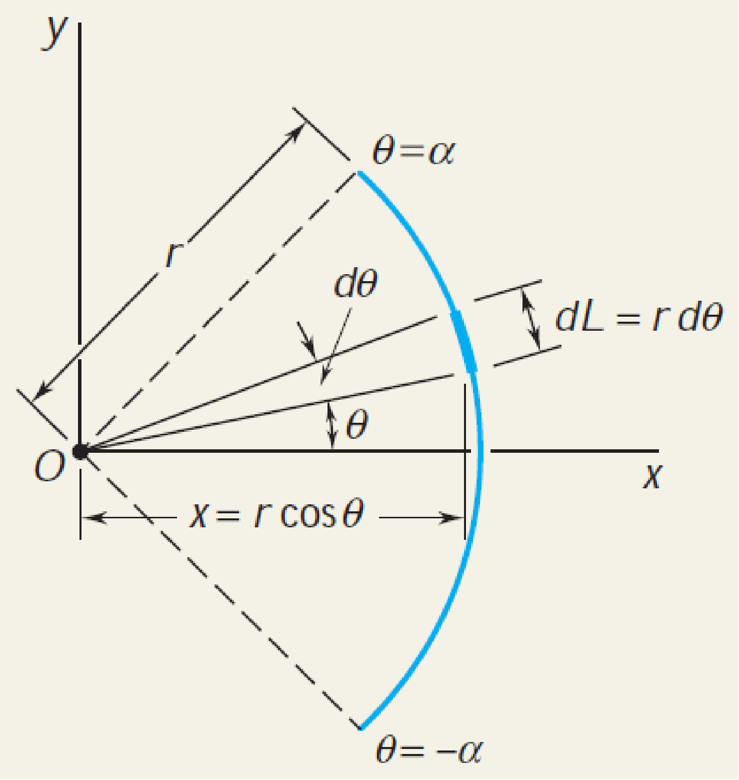
Detemline the location of the centroid of the circular arc shown.



❑ Sample Problem 04

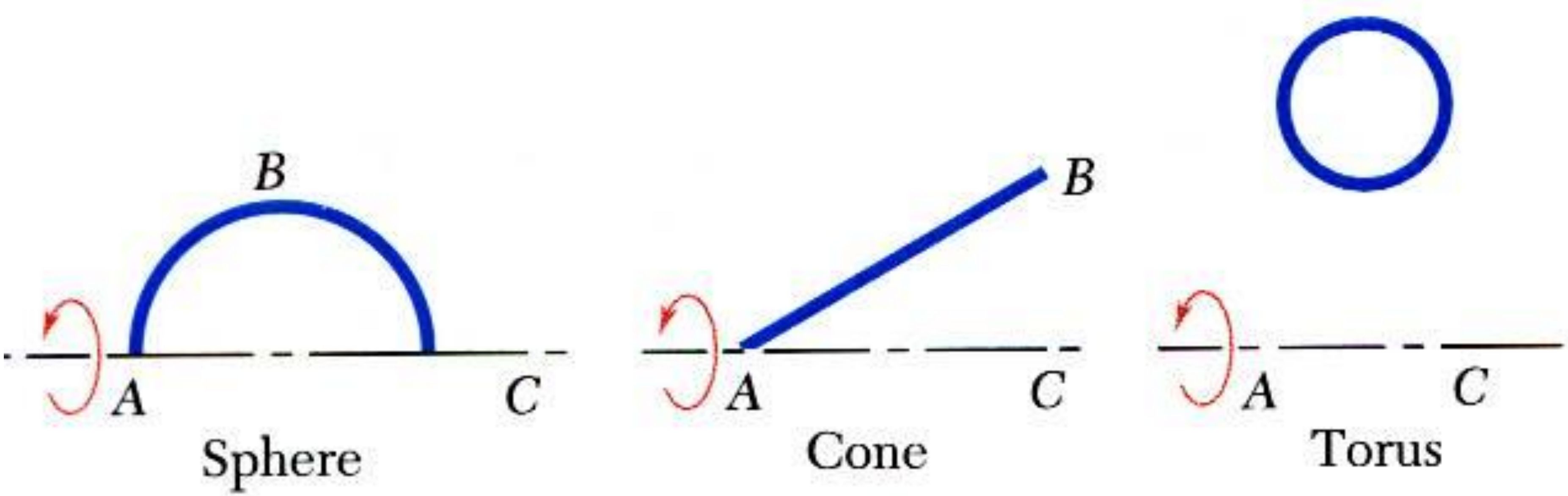
SOLUTION:

- Since the arc is symmetrical with respect to the x axis, $\bar{y} = 0$. A differential element is chosen as shown, and the length or the arc i determined by integration



❑ Theorems of Pappus-Guldinus

- ***Surface of revolution*** is generated by rotating a plane curve about a fixed axis.

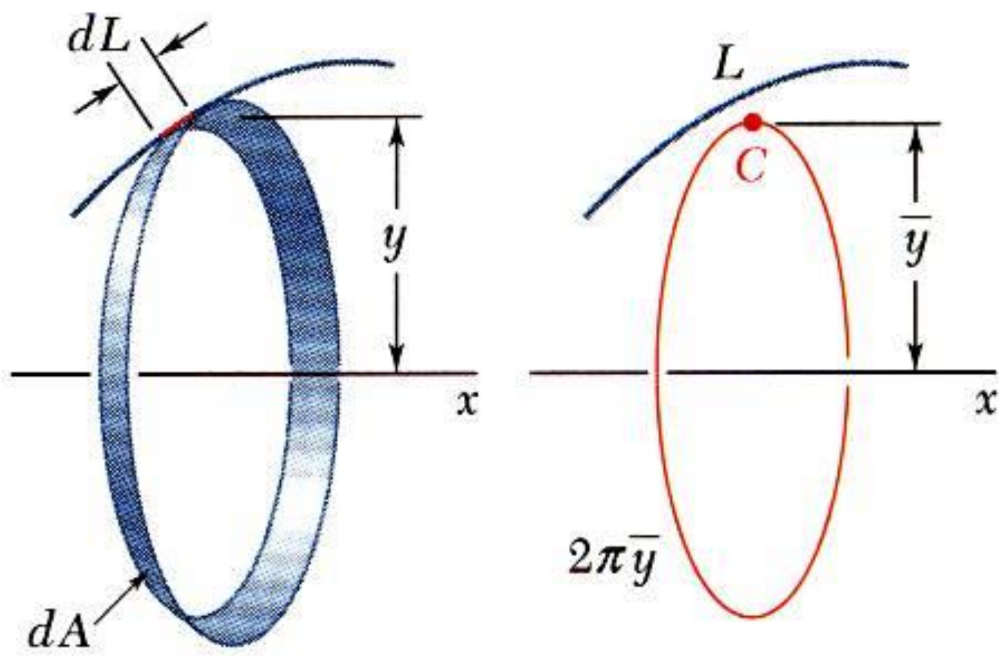


❑ Theorems of Pappus-Guldinus

- ***Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.***

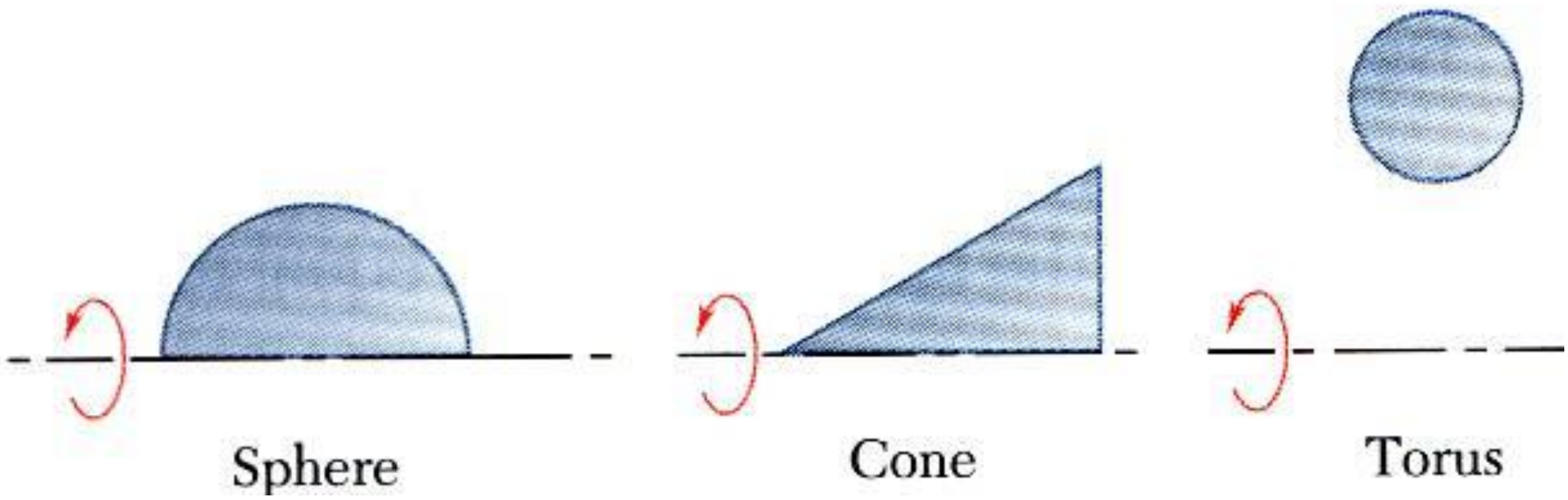
$$dA = 2\pi y\, dL \quad \Rightarrow \quad A = 2\pi \int y\, dL$$

$$\Rightarrow \boxed{A = 2\pi \bar{y} L}$$



Theorems of Pappus-Guldinus

- Body of revolution is generated by rotating a plane area about a fixed axis.

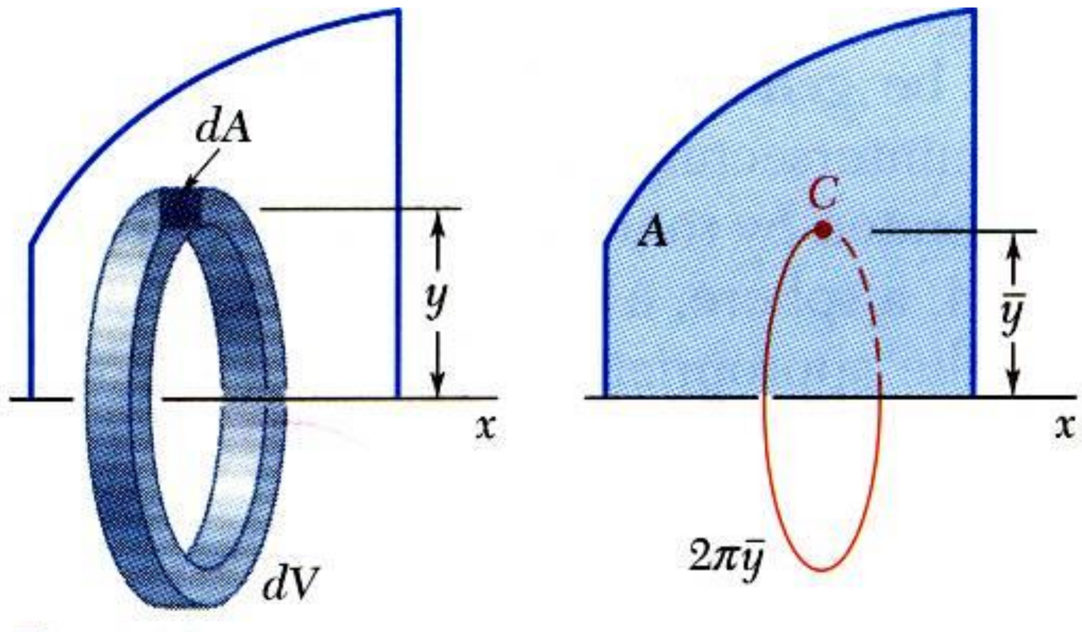


❑ Theorems of Pappus-Guldinus

- ***Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.***

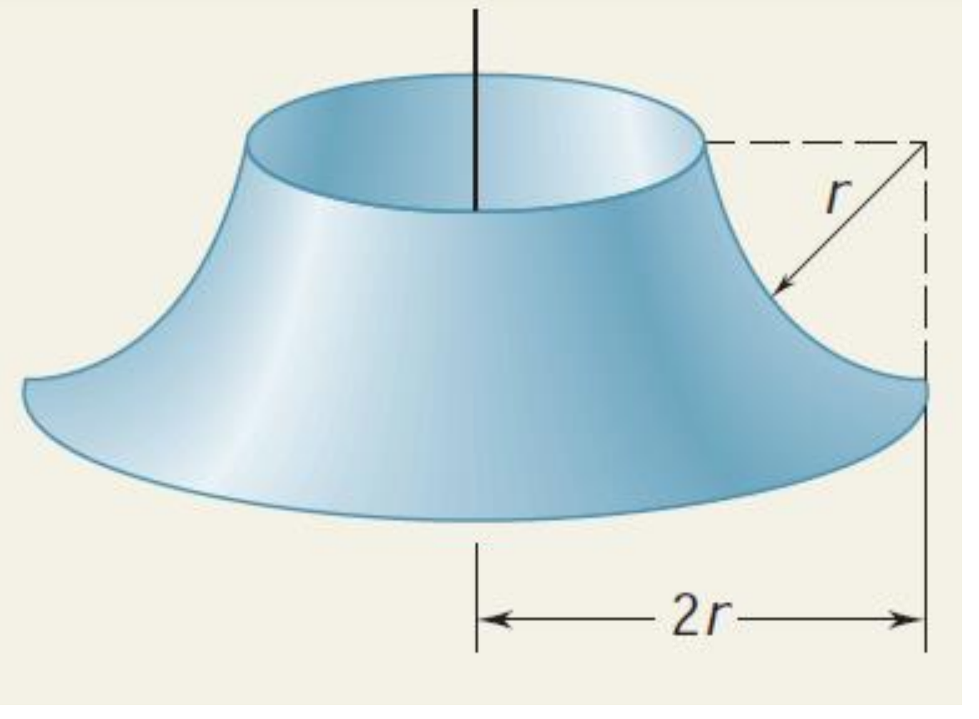
$$dV = 2\pi y\, dA \quad \Rightarrow \quad V = 2\pi \int y\, dA$$

$$\Rightarrow \boxed{V = 2\pi \bar{y}\, A}$$



❑ Sample Problem 05

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.



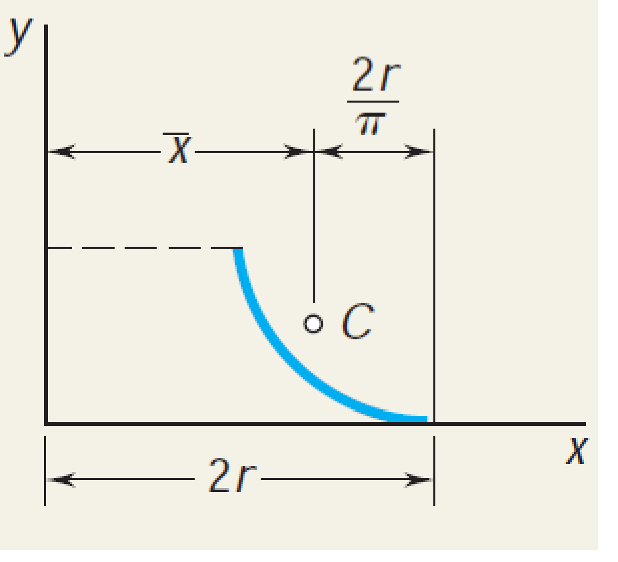
❑ Sample Problem 05

SOLUTION:

According to Theorem I of Pappu -Guldinus, the area generated is equal to the product of tlle length of the arc and Ule distance traveled by its centroid.

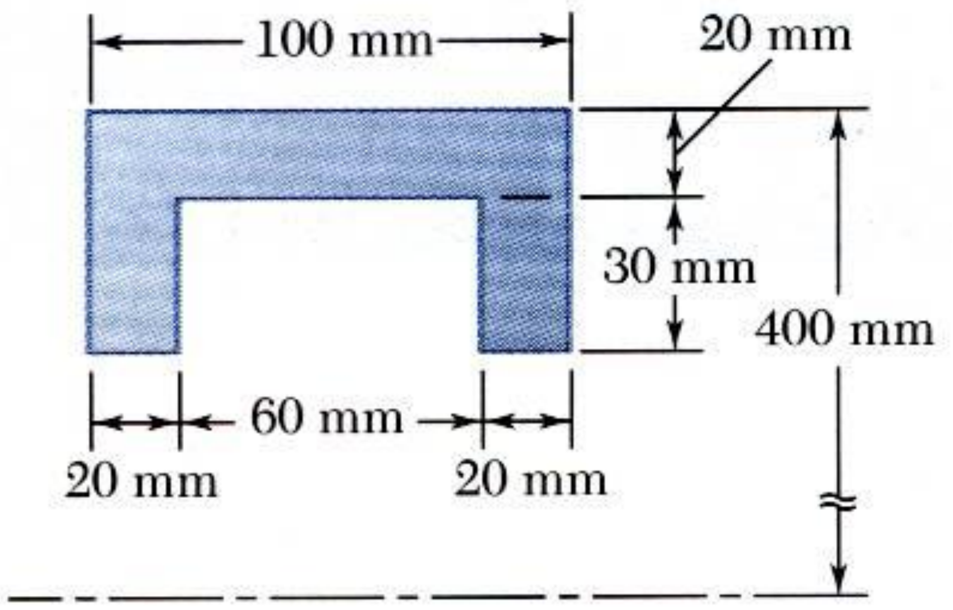
$$\bar{x} = 2r - \frac{2r}{\pi} \quad \Rightarrow \quad \boxed{\bar{x} = 2r\left(1 - \frac{1}{\pi}\right)}$$

$$A = 2\pi\,\bar{x}L = 2\pi\left[2r\left(1 - \frac{1}{\pi}\right)\right]\left(\frac{2\pi\, r}{4}\right) \quad \Rightarrow \quad \boxed{A = 2\pi\, r^2(\pi - 1)}$$



❑ Sample Problem 06

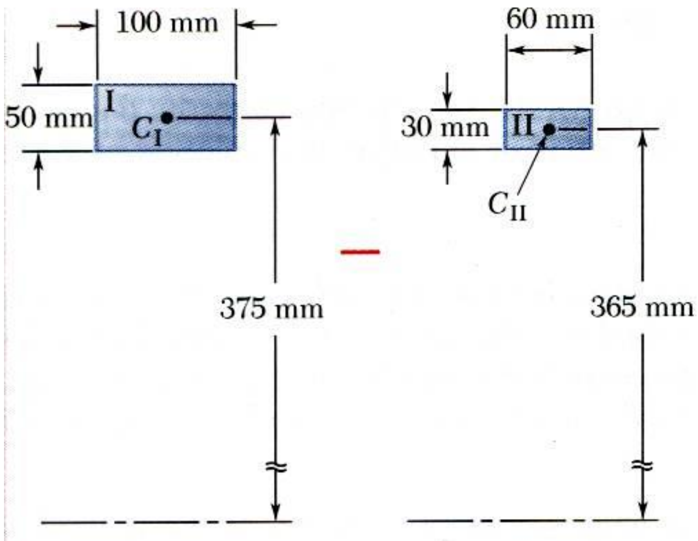
The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85\times10^3\ \mathrm{kg/m^3}$. determine the mass and weight of the rim.



Sample Problem 06

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.



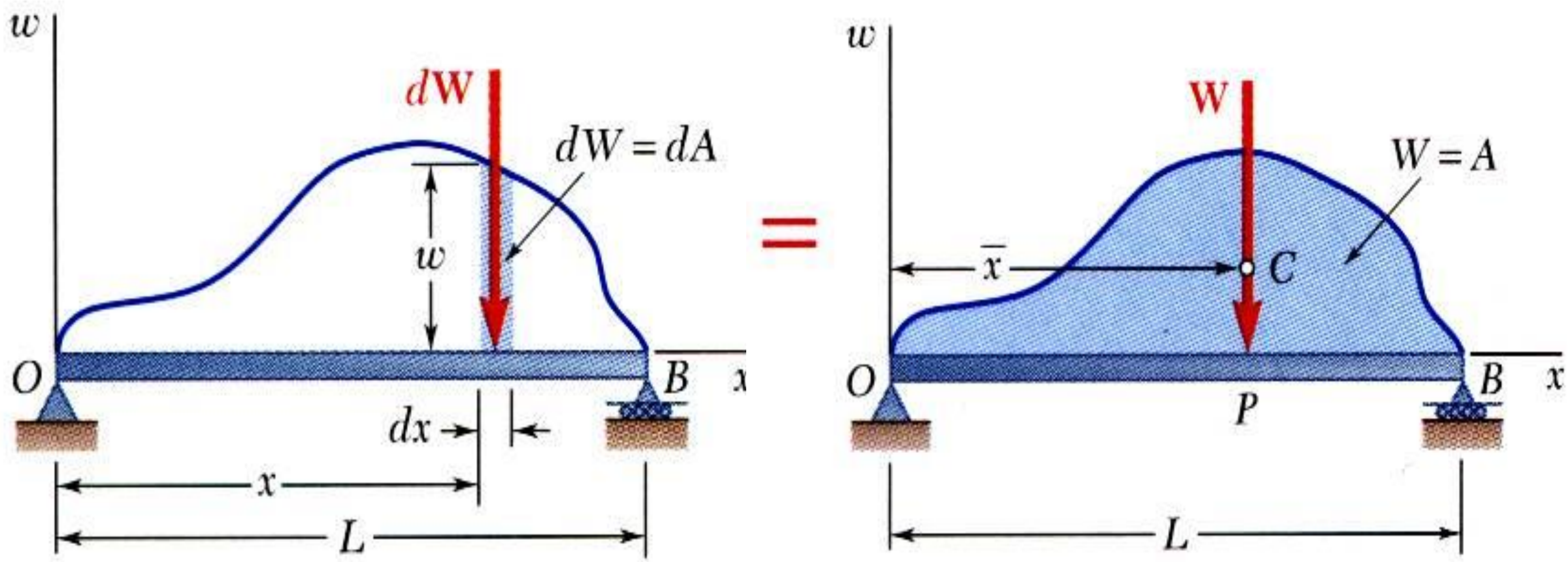
- Multiply by density and acceleration to get the mass and weight.

	Area, mm²	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm³
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

$$m = \rho V = (7.85\times10^3\,\text{kg/m}^3)\,(7.65\times10^6\,\text{mm}^3)\,(10^{-9}\,\text{m}^3/\text{mm}^3) \quad\Rightarrow\quad \boxed{m = 60.0\,\text{kg}}$$

$$W = mg = (60.0\,\text{kg})\,(9.81\,m/s^2) \quad\Rightarrow\quad \boxed{W = 589\,\text{N}}$$

Distributed Loads on Beams



- A distributed load is represented by plotting the load per unit length, w (N/m) . The total load is equal to the area under the load curve.

$$W = \int_0^L w\,dx = \int dA \quad \Rightarrow \quad \boxed{W = A}$$

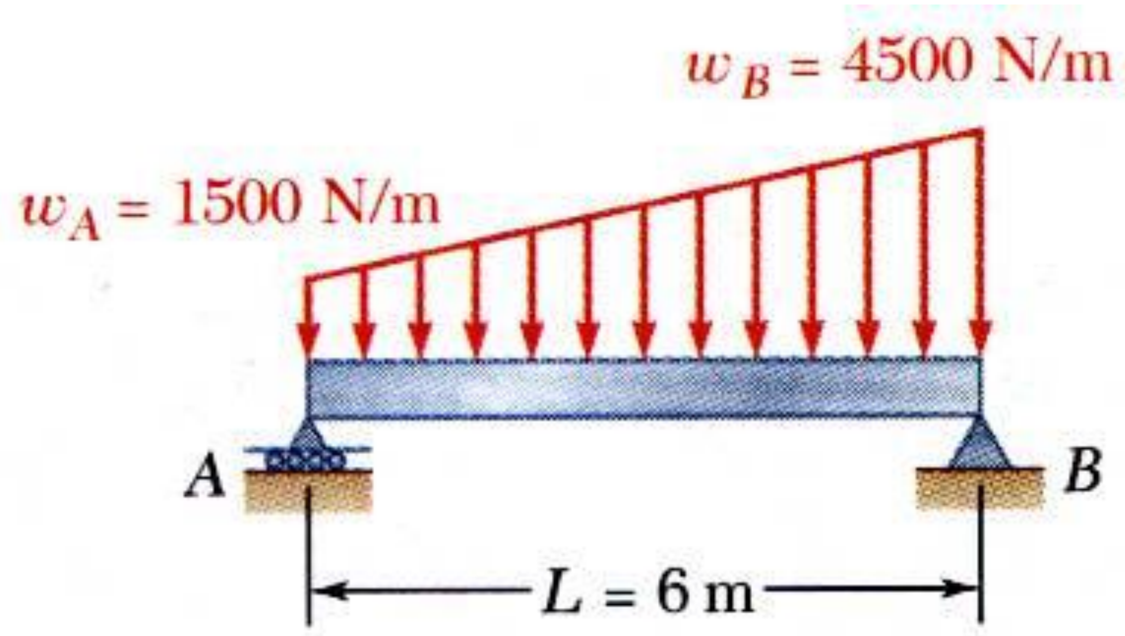
- ***A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.***

$$(OP)W = \int x\,dW \Rightarrow$$

$$(OP)\,A = \int_0^L x\,dA = \bar{x}A \quad \Rightarrow \quad \boxed{(OP) = \bar{x}}$$

Sample Problem 07

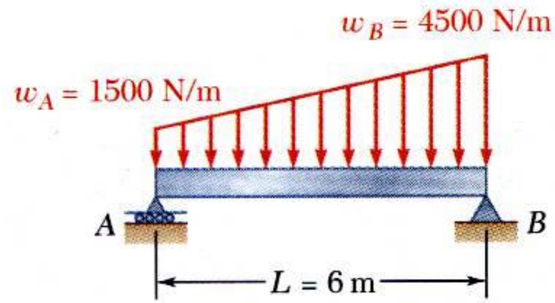
A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

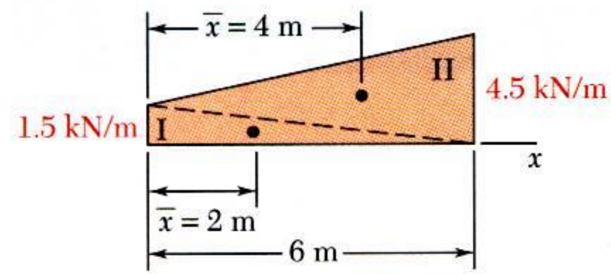


❑ Sample Problem 07

SOLUTION:

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN·m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$



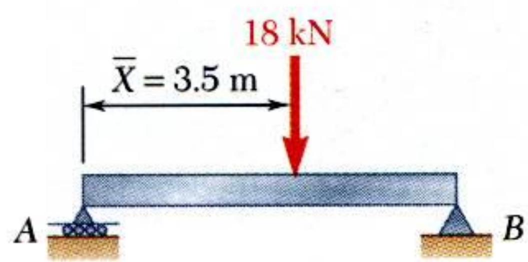


- The magnitude of the concentrated load is equal to the total load or the area under the curve.

$$F = \sum A \quad \Rightarrow \quad \boxed{F = 18.0\,\text{kN}}$$

- The line of action of the concentrated load passes through the centroid of the area under the curve.

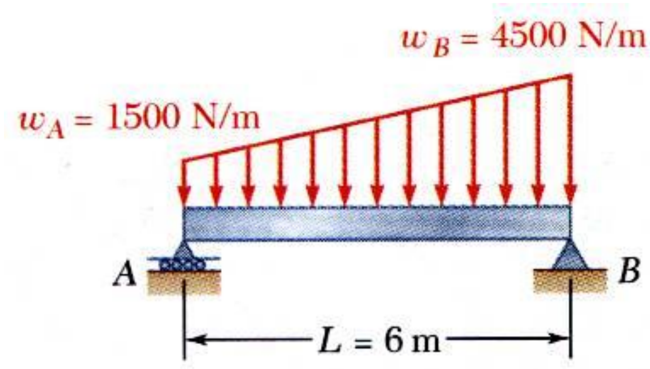
$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{63\,\text{kN}\cdot\text{m}}{18\,\text{kN}} \quad \Rightarrow \quad \boxed{\bar{X} = 3.5\,\text{m}}$$



Sample Problem 07

SOLUTION:

- Determine the support reactions by summing moments about the beam ends.

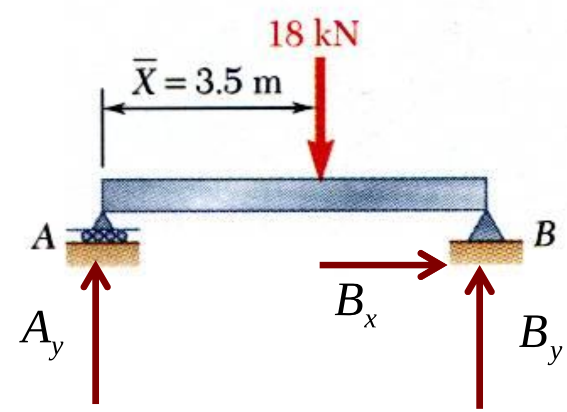


$$\sum F_x = 0 \quad \Rightarrow \quad \boxed{B_x = 0}$$

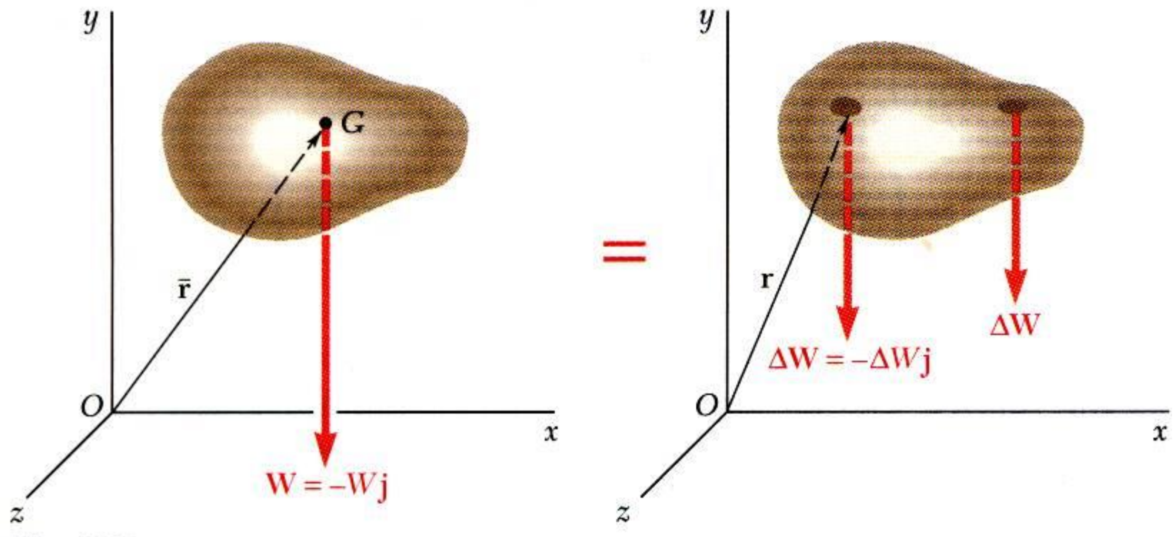
$$\sum M_A = 0: \quad B_y(6) - (18)(3.5) = 0 \quad \Rightarrow \quad \boxed{B_y = 10.5\,(\text{kN})}$$

$$\sum F_y = 0 \quad \Rightarrow \quad A_y + B_y - 18 = 0 \quad \Rightarrow \quad A_y + 10.5 - 18 = 0$$

$$\Rightarrow \quad \boxed{A_y = 7.5\,(\text{kN})}$$



❑ Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity G

$$-W\vec{j} = \sum(-\Delta W\vec{j})$$

$$\vec{r}_G \times (-W\vec{j}) = \sum[\vec{r} \times (-\Delta W\vec{j})]$$

$$\vec{r}_G W \times (-\vec{j}) = (\sum \vec{r}\Delta W) \times (-\vec{j})$$

$$W = \int dW \quad \Rightarrow \quad \boxed{\vec{r}_G W = \int \vec{r}\, dW}$$

- Results are independent of body orientation,

$$\boxed{\bar{x}W = \int x dW \quad \bar{y}W = \int y dW \quad \bar{z}W = \int z dW}$$

- For homogeneous bodies,

$$W = \gamma V \text{ and } dW = \gamma dV \quad \Rightarrow$$

$$\boxed{\bar{x}V = \int x dV \quad \bar{y}V = \int y dV \quad \bar{z}V = \int z dV}$$

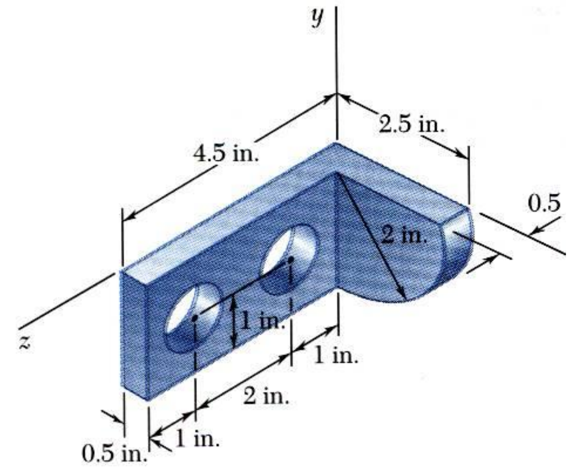
Composite 3D Bodies

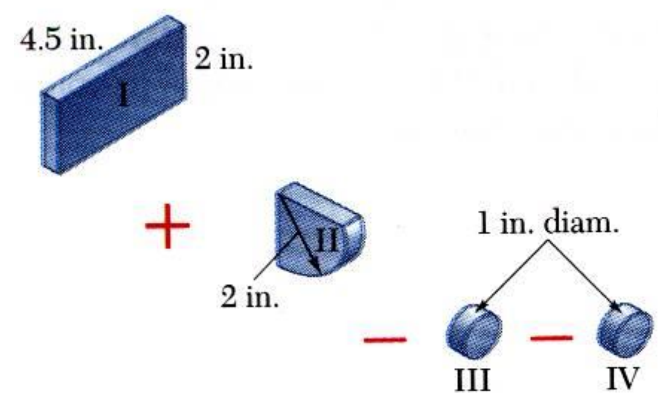
- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

- For homogeneous bodies,

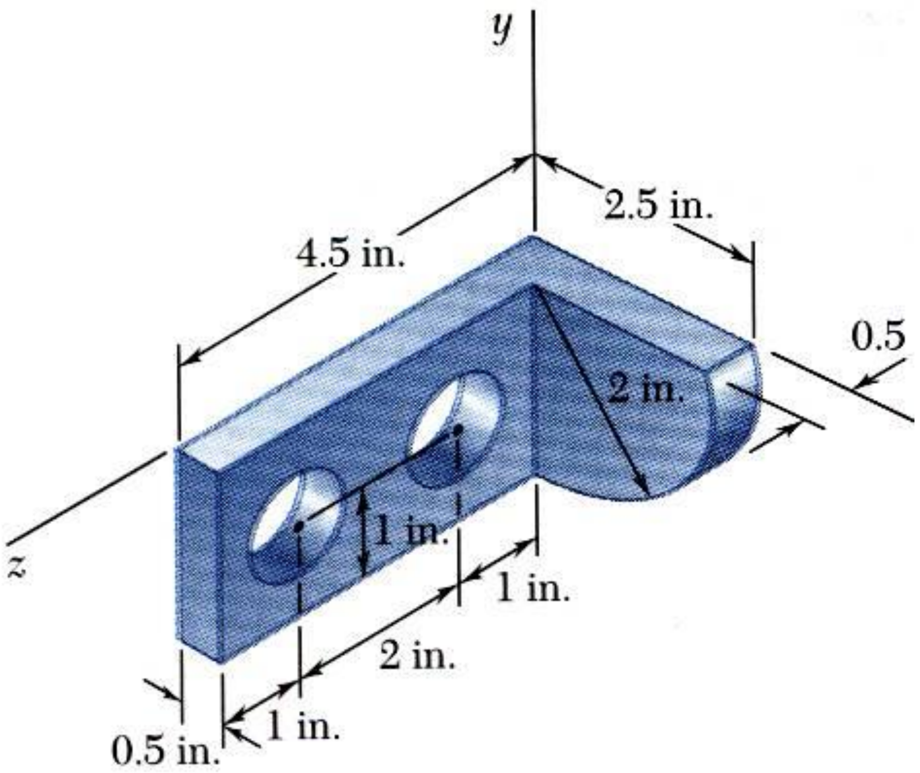
$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$





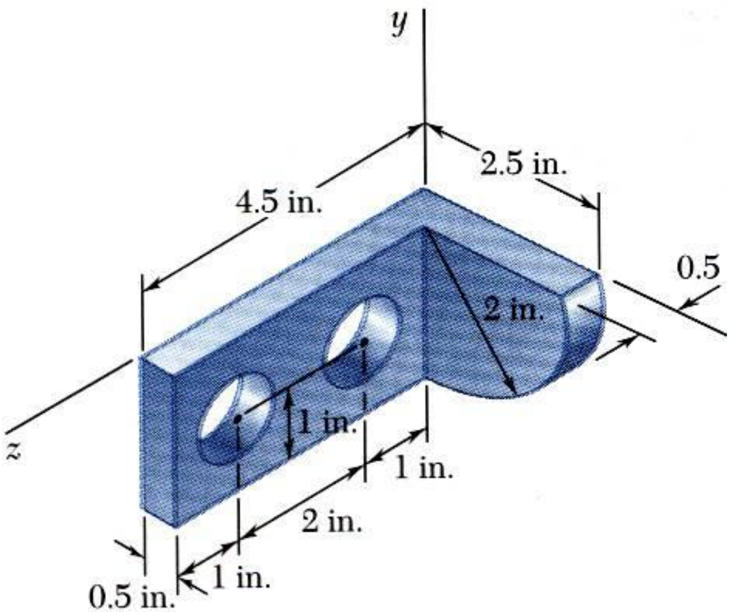
❑ Sample Problem 08

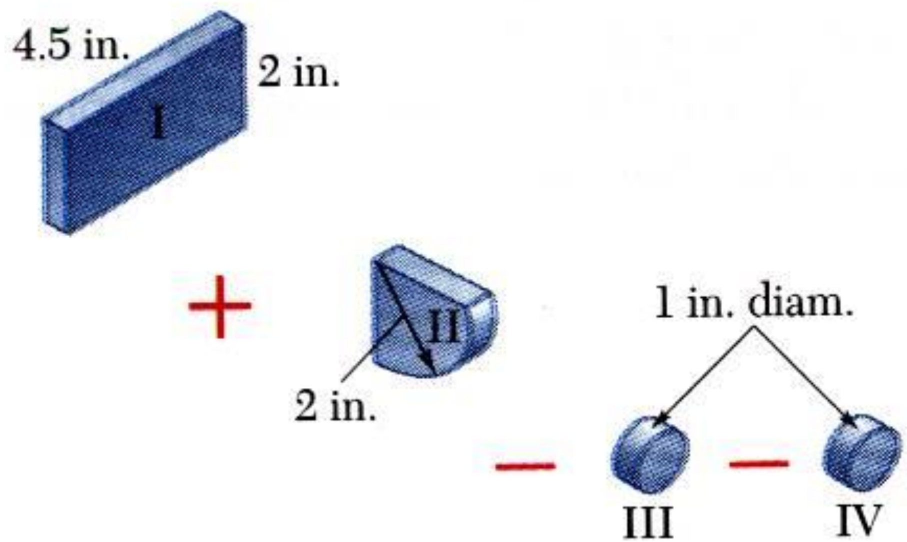
Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.



❑ Sample Problem 08

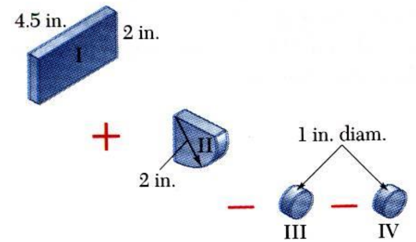
SOLUTION:

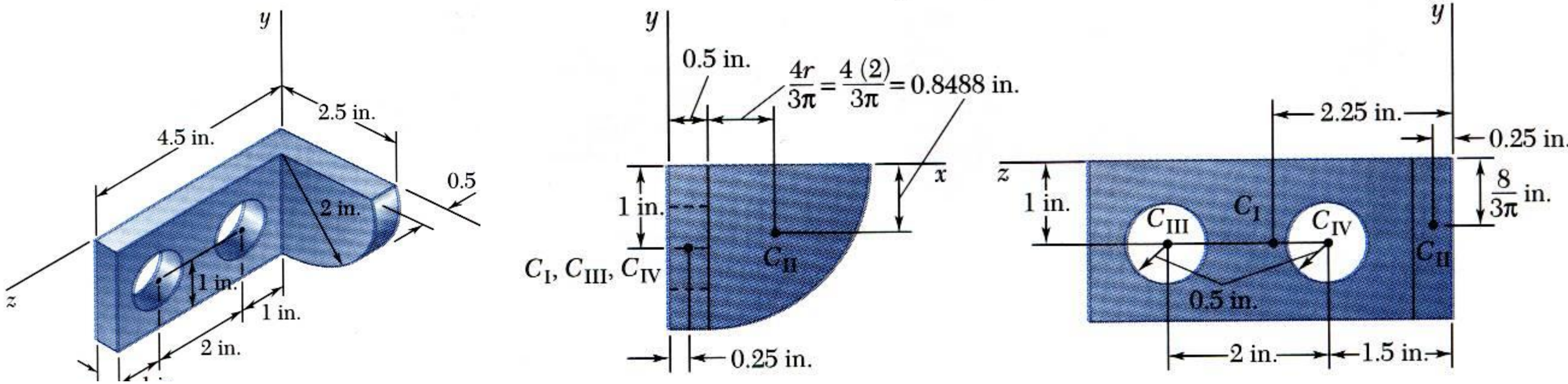




❑ Sample Problem 08

SOLUTION:





	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

❑ Sample Problem 08

SOLUTION:

	V, in^3	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in^4	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma \bar{x}V = 3.048$	$\Sigma \bar{y}V = -5.047$	$\Sigma \bar{z}V = 8.555$

$$\bar{X} = \frac{\sum \bar{x}V}{\sum V} = \frac{(3.08\,\text{in}^4)}{(5.286\,\text{in}^3)} \Rightarrow \boxed{\bar{X} = 0.577\,\text{in.}}$$

$$\bar{Y} = \frac{\sum \bar{y}V}{\sum V} = \frac{(-5.047\,\text{in}^4)}{(5.286\,\text{in}^3)} \Rightarrow \boxed{\bar{Y} = 0.577\,\text{in.}}$$

$$\bar{Z} = \frac{\sum \bar{z}V}{\sum V} = \frac{(1.618\,\text{in}^4)}{(5.286\,\text{in}^3)} \Rightarrow \boxed{\bar{Z} = 0.577\,\text{in.}}$$