Chapter 7: Centroids and Centers of Gravity

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□ Introduction

- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.
- The *centroid* of an area is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.
- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

- **Center of Gravity of a 2D Body**
 - Center of gravity of a plate



$$\sum F_{z}: \implies W = \Delta W_{1} + \Delta W_{1} + \dots + \Delta W_{n}$$

$$\sum M_{y}: \implies \overline{x}W = \sum x\Delta W \implies \overline{x} = \frac{\sum x\Delta W}{W} \quad or \quad \overline{x} = \frac{\int x\,dW}{W}$$

$$\sum M_{x}: \implies \overline{y}W = \sum y\Delta W \implies \overline{y} = \frac{\sum y\Delta W}{W} \quad or \quad \overline{y} = \frac{\int y\,dW}{W}$$

Center of Gravity of a 2D Body

• Center of gravity of a wire



$$\sum M_{y}: \implies \bar{x}W = \sum x\Delta W \implies$$

$$\left(\overline{x} = \frac{\sum x \Delta W}{W} \quad or \quad \overline{x} = \frac{\int x \, dW}{W}\right)$$

$$\sum M_x: \implies \overline{y}W = \sum y\Delta W \implies$$

$$\overline{y} = \frac{\sum y \Delta W}{W} \quad or \quad \overline{y} = \frac{\int y \, dW}{W}$$

Centroids and First Moments of Areas

• Centroid of an area



$$\Delta W = \gamma t \, \Delta A \quad \Longrightarrow \qquad W = \gamma t \, A$$

$$\overline{x} = \frac{\int x \, dW}{W} \implies \overline{x} = \frac{\int x(\gamma t) dA}{(\gamma A t)} \implies \left(\overline{x} = \frac{\int x \, dA}{A}\right)$$

$$\overline{y} = \frac{\int y \, dW}{W} \implies \overline{y} = \frac{\int y(\gamma t) dA}{(\gamma A t)} \implies \overline{\overline{y}} = \frac{\int y \, dA}{A}$$

if
$$Q_y = \int x \, dA \implies Q_y = \bar{x}A$$

 Q_y : First moment with respect to y axis

$$if \quad \left[Q_x = \int y \, dA \quad \Rightarrow \qquad Q_x = \overline{y}A \right]$$

 Q_x : First moment with respect to x axis

Centroids and First Moments of Lines

y

• Centroid of a line

$$\Delta W = \gamma \, a \, \Delta L \quad \Rightarrow \boxed{W = \gamma \, a \, L}$$

a: Cross section area

$$\bar{x} = \frac{\int x \, dW}{W} \implies \bar{x} = \frac{\int x(\gamma a) \, dL}{(\gamma L a)} \implies \left(\bar{x} = \frac{\int x \, dL}{L} \right)$$
$$\bar{y} = \frac{\int y \, dW}{W} \implies \bar{y} = \frac{\int y(\gamma a) \, dL}{(\gamma L a)} \implies \left(\bar{y} = \frac{\int y \, dL}{L} \right)$$

$$y$$

 $x \rightarrow \Delta L$
 y
 Q

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$
$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

x

□ First Moments of Areas and Lines

• An area is symmetric with respect to an axis *BB*' if for every point *P* there exists a point *P*' such that *PP*' is perpendicular to *BB*' and is divided into two equal parts by *BB*'.

- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis





□ First Moments of Areas and Lines

• If an area possesses two lines of symmetry, its centroid lies at their intersection.

- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA*' of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.



Centroids of Common Shapes of Areas

Shape	THE DESCRIPTION OF THE PARTY OF	x	\overline{y}	Area
Triangular area	$\frac{1}{ \overline{y} } \xrightarrow{c} \xrightarrow{h} \xrightarrow{h} \xrightarrow{h} \xrightarrow{h} \xrightarrow{h} \xrightarrow{h} \xrightarrow{h} h$		$\frac{h}{3}$	<u>bh</u> 2
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area	$\frac{1}{\overline{x}}$	0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area	$ \begin{array}{c} 0 \\ \rightarrow \end{array} \xrightarrow{\ddagger y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\ddagger y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow x} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow x} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow x} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \hline \end{array} \begin{array}{c} 0 \\ \hline \end{array} \xrightarrow{\downarrow y} \\ \end{array} \end{array}$	0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area	$0 \xrightarrow{\overline{y}} 0 \xrightarrow{h} +$	0	$\frac{3h}{5}$	$\frac{4ah}{3}$

Centroids of Common Shapes of Areas

Parabolic spandrel	$a \qquad y = kx^2 \qquad h \\ \hline \\$	$\frac{3a}{4}$	$\frac{3h}{10}$	<u>ah</u> 3
General spandrel	$O = \frac{x}{\overline{x}}$	$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$.	0	$lpha r^2$

Centroids of Common Shapes of Lines

Shape		x	\overline{y}	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc	$0 \frac{y}{x} \frac{y}{0} \frac{r}{1 - r}{1 - \frac{r}{1 - r}{1 - \frac{r}{1 - r}{1 - \frac{r}{1 - r}}}}}}}}}$	0	$\frac{2r}{\pi}$	πr
Arc of circle	r α c α \overline{x}	$\frac{r \sin \alpha}{\alpha}$	0	2ar

Composite Plates

• Composite plates





Composite Areas

• Composite area





Composite Areas

• Composite area



	X	Α	πA
A ₁ Semicircle		+	-
A ₂ Full rectangle	+	+	+
A ₃ Circular hole	+	_	—

For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.



SOLUTION:



Component	A, mm ²	⊼, mm	ӯ, mm	⊼A, mm³	ӯA, mm³
Rectangle Triangle Semicircle Circle	$\begin{array}{l} (120)(80) = 9.6 \times 10^3 \\ \frac{1}{2}(120)(60) = 3.6 \times 10^3 \\ \frac{1}{2}\pi(60)^2 = 5.655 \times 10^3 \\ -\pi(40)^2 = -5.027 \times 10^3 \end{array}$	60 40 60 60	$40 \\ -20 \\ 105.46 \\ 80$	$\begin{array}{r} +576 \times 10^{3} \\ +144 \times 10^{3} \\ +339.3 \times 10^{3} \\ -301.6 \times 10^{3} \end{array}$	$\begin{array}{r} +384 \times 10^{3} \\ -72 \times 10^{3} \\ +596.4 \times 10^{3} \\ -402.2 \times 10^{3} \end{array}$
	$\Sigma A = 13.828 \times 10^3$			$\Sigma \bar{x}A = +757.7 \times 10^3$	$\Sigma \overline{y}A = +506.2 \times 10^3$

• Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2 \times 10^3 \,\mathrm{mm^3}$$
$$Q_y = +757.7 \times 10^3 \,\mathrm{mm^3}$$

SOLUTION:

• Compute the coordinates of the area centroid by dividing the first moments by the total area.

$$\overline{X} = \frac{\sum \overline{x}A}{\sum A} = \frac{+757.7 \times 10^3 \,\mathrm{mm}^3}{13.828 \times 10^3 \,\mathrm{mm}^2} \implies \overline{X} = 54.8 \,(mm)$$

$$\overline{Y} = \frac{\sum \overline{y}A}{\sum A} = \frac{+506.2 \times 10^3 \,\mathrm{mm}^3}{13.828 \times 10^3 \,\mathrm{mm}^2} \implies \overline{Y} = 36.6 \,(mm)$$

x

The figure shown is made from a piece of thin, homogeneoius wire Determine the location of its center of gravity.



Solution: Solution:



Segment	<i>L</i> , in.	\overline{x} , in.	\overline{y} , in.	$\overline{x}L$, in ²	$\overline{y}L$, in ²
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
	$\Sigma L = 60$	18.2		$\Sigma \overline{x}L = 600$	$\Sigma \overline{y}L = 180$

$$\overline{X} = \frac{\sum \overline{x}L}{\sum L} = \frac{600}{60} \implies (\overline{X} = 10 \text{ (in.)}) \qquad \overline{Y} = \frac{\sum \overline{y}L}{\sum L} = \frac{180}{60} \implies (\overline{Y} = 3 \text{ (in.)})$$

Determination of Centroids by Integration

• Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.

$$Q_{y} = \bar{x}A = \int xdA \implies Q_{y} = \iint x\,dx\,dy \quad or \quad Q_{y} = \int \bar{x}_{el}\,dA$$

$$Q_{x} = \bar{y}A = \int ydA \implies Q_{x} = \iint y\,dx\,dy \quad or \quad Q_{x} = \int \bar{y}_{el}\,dA$$

$$Q_{y} = \bar{x}A = \int \bar{x}_{el}\,dA \implies Q_{y} = \int x\,(ydx)$$

$$Q_{x} = \bar{y}A = \int \bar{y}_{el}\,dA \implies Q_{x} = \int \frac{y}{2}(ydx)$$

Determination of Centroids by Integration

$$Q_{y} = \bar{x}A = \int \bar{x}_{el} dA \implies Q_{y} = \int \frac{a+x}{2} [(a-x) dy]$$

$$Q_{x} = \bar{y}A = \int \bar{y}_{el} dA \implies Q_{x} = \int y[(a-x) dy]$$

$$Q_{x} = \bar{y}A = \int \bar{y}_{el} dA \implies Q_{x} = \int y[(a-x) dy]$$

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Determination of Centroids by Integration

$$Q_{y} = \bar{x}A = \int \bar{x}_{el} dA \implies Q_{y} = \int \frac{2r}{3} \cos \theta \left(\frac{1}{2}r^{2}d\theta\right)$$

$$Q_{x} = \bar{y}A = \int \bar{y}_{el} dA \implies Q_{x} = \int \frac{2r}{3} \sin \theta \left(\frac{1}{2}r^{2}d\theta\right)$$

$$Q_{x} = \bar{y}A = \int \bar{y}_{el} dA \implies Q_{x} = \int \frac{2r}{3} \sin \theta \left(\frac{1}{2}r^{2}d\theta\right)$$

Determine by direct integration the location of the centroid of a parabolic spandrel.



SOLUTION:

• Determine the constant k.



SOLUTION:

• Using vertical strips, perform a single integration to find the first moments.



SOLUTION:

• Or, using horizontal strips, perform a single integration to find the first moments.



SOLUTION:

• Evaluate the centroid coordinates.



Detemline the location of the centroid of the circular arc shown.



Distributed Forces: Centroids and Centers of Gravity Sample Problem 04

SOLUTION:

• Since the arc is symmetrical with respect to the x axis, $\bar{y} = 0$. A differential element is chosen as shown, and the length or the arc i determined by integration



Distributed Forces: Centroids and Centers of Gravity

Theorems of Pappus-Guldinus

• *Surface of revolution* is generated by rotating a plane curve about a fixed axis.



Theorems of Pappus-Guldinus

• Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$dA = 2\pi y \, dL \implies A = 2\pi \int y \, dL$$
$$\implies A = 2\pi \overline{y}L$$

dL

y

x

Theorems of Pappus-Guldinus

• Body of revolution is generated by rotating a plane area about a fixed axis.



Theorems of Pappus-Guldinus

• Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$dV = 2\pi y \, dA \implies V = 2\pi \int y \, dA$$
$$\Rightarrow \boxed{V = 2\pi \overline{y} A}$$

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.



SOLUTION:

According to Theorem I of Pappu -Guldinus, the area generated is equal to the product of tlle length of the arc and Ule distance traveled by its centroid.

VI

$$\bar{x} = 2r - \frac{2r}{\pi} \implies \left(\bar{x} = 2r \left(1 - \frac{1}{\pi} \right) \right)$$

$$A = 2\pi \bar{x}L = 2\pi \left[2r \left(1 - \frac{1}{\pi} \right) \right] \left(\frac{2\pi r}{4} \right) \implies \left(A = 2\pi r^2 (\pi - 1) \right)$$

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85 \times 10^3 \text{ kg/m}^3$. determine the mass and weight of the rim.



Distributed Forces: Centroids and Centers of Gravity

Sample Problem 06

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.
- Multiply by density and acceleration to get the mass and weight.



	Area, mm ²	ӯ, mm	Distance Traveled by <i>C</i> , mm	Volume, mm ³
I II	$+5000 \\ -1800$	375 365	$2\pi(375) = 2356$ $2\pi(365) = 2293$	$(5000)(2356) = 11.78 \times 10^{6}$ $(-1800)(2293) = -4.13 \times 10^{6}$
	5		e in più e (1919 - 1919)	Volume of rim = 7.65×10^6

 $m = \rho V = (7.85 \times 10^3 \text{ kg/m}^3) (7.65 \times 10^6 \text{ mm}^3) (10^{-9} \text{ m}^3 / \text{ mm}^3) \implies (m = 60.0 \text{ kg})$

 $W = mg = (60.0 \text{ kg}) (9.81 \text{ m/s}^2) \implies W = 589 \text{ N}$

Distributed Loads on Beams



• A distributed load is represented by plotting the load per unit length, w (N/m). The total load is equal to the area under the load curve.

 $W = \int_{0}^{L} w dx = \int dA \quad \Longrightarrow \qquad W = A$

• A distributed load can be replace by a concentrated load $(OP)W = \int x dW \Rightarrow$ with a magnitude equal to the area under the load curve and a line of action passing through the area centroid. $(OP)A = \int_{0}^{L} x dA = \bar{x}A \Rightarrow (OP) = \bar{x}$

A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.



or the area under the curve.

SOLUTION:

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Component	A, kN	x , m	<i>⊼A</i> , kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

The magnitude of the concentrated load is equal to the total load



- 1.5 kN/m $\overline{x} = 2 \text{ m}$ $\overline{x} = 2 \text{ m}$ $\overline{x} = 2 \text{ m}$
- The line of action of the concentrated load passes through the centroid of the area under the curve.

 $F = \sum A \implies | F = 18.0 \text{ kN}$

$$\overline{X} = \frac{\sum \overline{x}A}{\sum A} = \frac{63 \text{ kN} \cdot \text{m}}{18 \text{ kN}} \implies \overline{X} = 3.5 \text{ m}$$



SOLUTION:

• Determine the support reactions by summing moments about the beam ends.

$$\sum F_x = 0 \quad \Rightarrow \quad B_x = 0$$

$$\sum M_A = 0$$
: $B_y(6) - (18)(3.5) = 0 \implies B_y = 10.5 \text{ (kN)}$

$$\sum F_{y} = 0 \implies A_{y} + B_{y} - 18 = 0 \implies A_{y} + 10.5 - 18 = 0$$
$$\implies A_{y} = 7.5 \text{ (kN)}$$





Center of Gravity of a 3D Body: Centroid of a Volume



• Center of gravity G $-W \vec{j} = \sum (-\Delta W \vec{j})$

$$\vec{r}_{G} \times (-W \,\vec{j}) = \sum [\vec{r} \times (-\Delta W \,\vec{j})]$$
$$\vec{r}_{G} W \times (-\vec{j}) = (\sum \vec{r} \Delta W) \times (-\vec{j})$$
$$W = \int dW \implies \overrightarrow{r}_{G} W = \int \vec{r} dW$$

- Results are independent of body orientation, $\overline{x}W = \int x dW \quad \overline{y}W = \int y dW \quad \overline{z}W = \int z dW$
- For homogeneous bodies,

$$W = \gamma V$$
 and $dW = \gamma dV \implies$

$$\overline{x}V = \int x dV \quad \overline{y}V = \int y dV \quad \overline{z}V = \int z dV$$

Composite 3D Bodies

• Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\overline{X}\sum W = \sum \overline{x}W \quad \overline{Y}\sum W = \sum \overline{y}W \quad \overline{Z}\sum W = \sum \overline{z}W$$

• For homogeneous bodies,

$$\overline{X}\sum V = \sum \overline{x}V \quad \overline{Y}\sum V = \sum \overline{y}V \quad \overline{Z}\sum V = \sum \overline{z}V$$



Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in. y_{\parallel}



SOLUTION:





SOLUTION:

	V, in ³	⊼ , in.	<u></u> <i>y</i> , in.	, in.	$\overline{x}V$, in ⁴	ӯ <i>V</i> , in⁴	Σ <i>V</i> , in⁴
I II III IV	$\begin{array}{l} (4.5)(2)(0.5) = 4.5\\ \frac{1}{4}\pi(2)^2(0.5) = 1.571\\ -\pi(0.5)^2(0.5) = -0.3927\\ -\pi(0.5)^2(0.5) = -0.3927 \end{array}$	$\begin{array}{c} 0.25 \\ 1.3488 \\ 0.25 \\ 0.25 \end{array}$	-1 -0.8488 -1 -1	$2.25 \\ 0.25 \\ 3.5 \\ 1.5$	1.125 2.119 -0.098 -0.098	-4.5 -1.333 0.393 0.393	$10.125 \\ 0.393 \\ -1.374 \\ -0.589$
	$\Sigma V = 5.286$				$\Sigma \overline{x}V = 3.048$	$\Sigma \overline{y}V = -5.047$	$\Sigma \overline{z} V = 8.555$

$$\overline{X} = \frac{\sum \overline{x}V}{\sum V} = \frac{(3.08 \text{ in}^4)}{(5.286 \text{ in}^3)} \implies \overline{X} = 0.577 \text{ in.}$$
$$\overline{Y} = \frac{\sum \overline{y}V}{\sum V} = \frac{(-5.047 \text{ in}^4)}{(5.286 \text{ in}^3)} \implies \overline{Y} = 0.577 \text{ in.}$$
$$\overline{Z} = \frac{\sum \overline{z}V}{\sum V} = \frac{(1.618 \text{ in}^4)}{(5.286 \text{ in}^3)} \implies \overline{Z} = 0.577 \text{ in.}$$