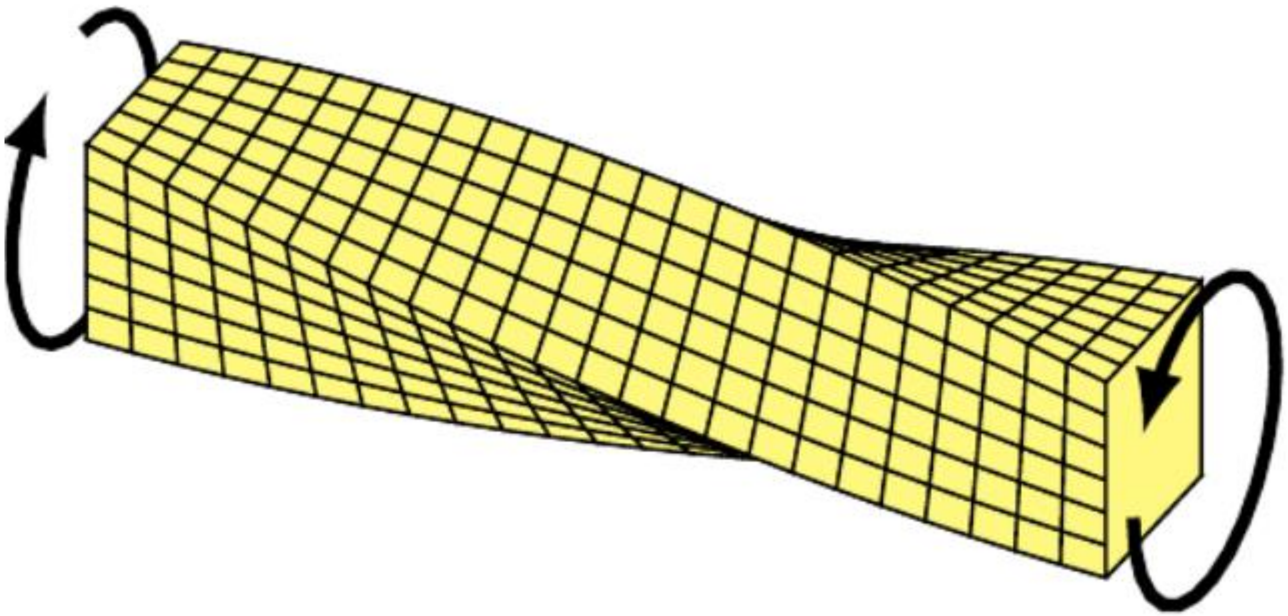
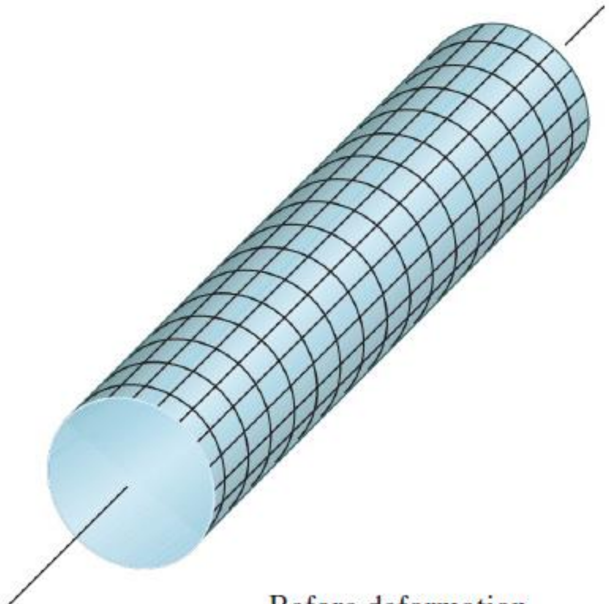
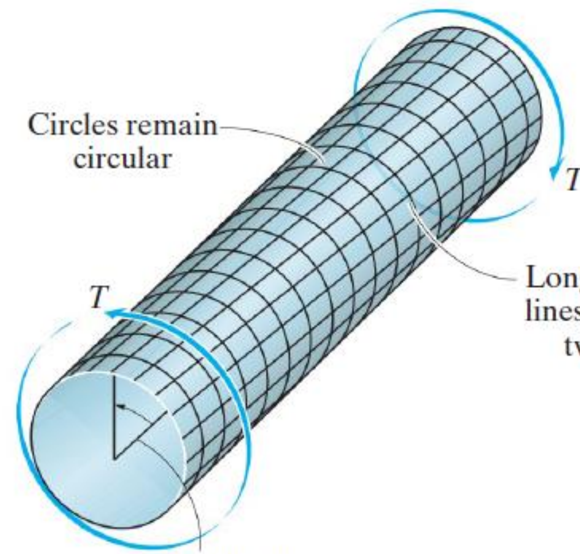
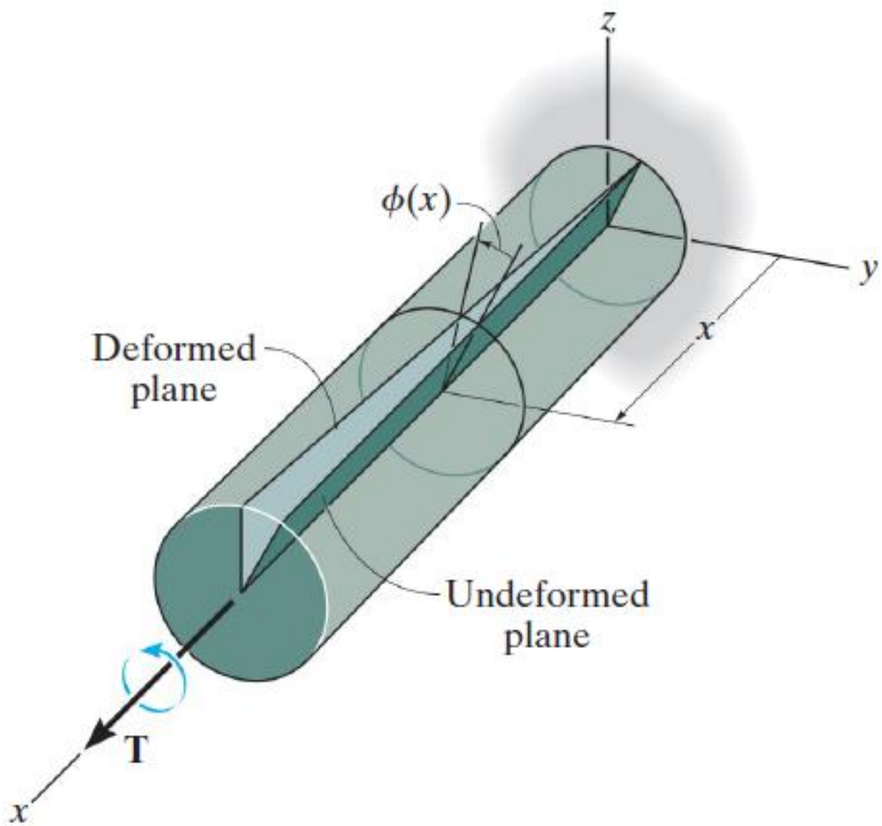
TORSION

فصل چهارم : پیچش



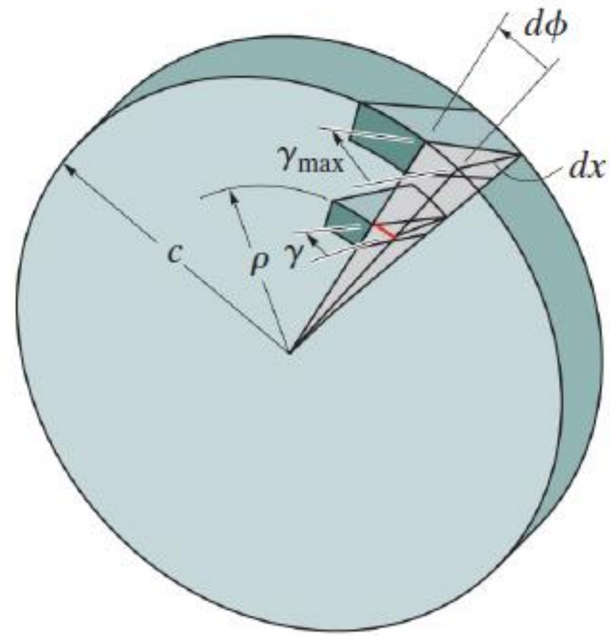

Before deformation
(a)


Circles remain circular
T
Longitudinal lines become twisted
T
Radial lines remain straight



The angle of twist $\phi(x)$ increases as x increases.

(a)



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

(b)

$$\rho \, d\phi = dx \, \gamma$$

$$\gamma = \rho \frac{d\phi}{dx}$$

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{max}$$

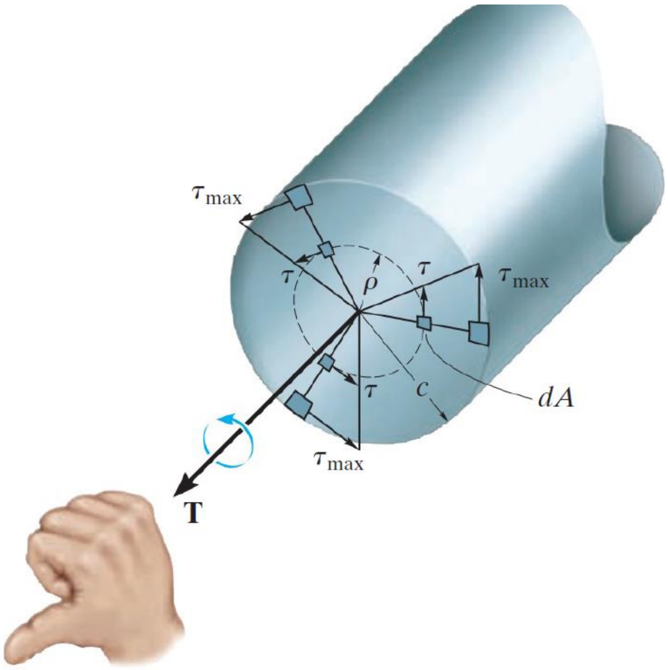
THE TORSION FORMULA

Hooke's law applies, $\tau = G\gamma$

$$\tau_{max} = G\gamma_{max}$$

linear variation in shear strain

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max}$$



Shear stress varies linearly along each radial line of the cross section.

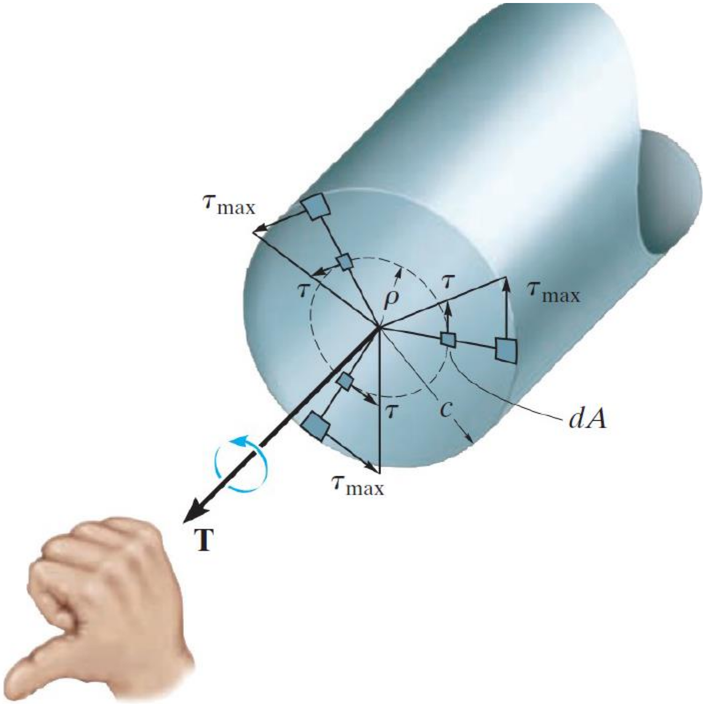
THE TORSION FORMULA

Since each element of area dA, located at ρ, is subjected to a force of $dF = \tau\, dA$, Fig. 5–3, the torque produced by this force is then $dT = \rho(\tau\, dA)$. For the entire cross section we have

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max}\, dA$$

However, $\tau_{\max}/c$ is constant, and so

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2\, dA$$



Shear stress varies linearly along each radial line of the cross section.

THE TORSION FORMULA

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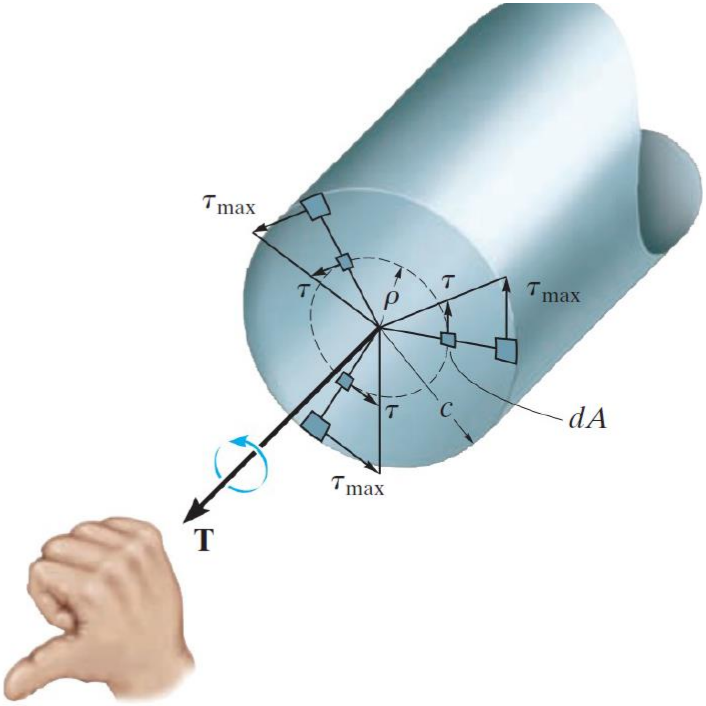


Shear stress varies linearly along each radial line of the cross section.

However, $\tau_{\max}/c$ is constant, and so

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2\, dA$$

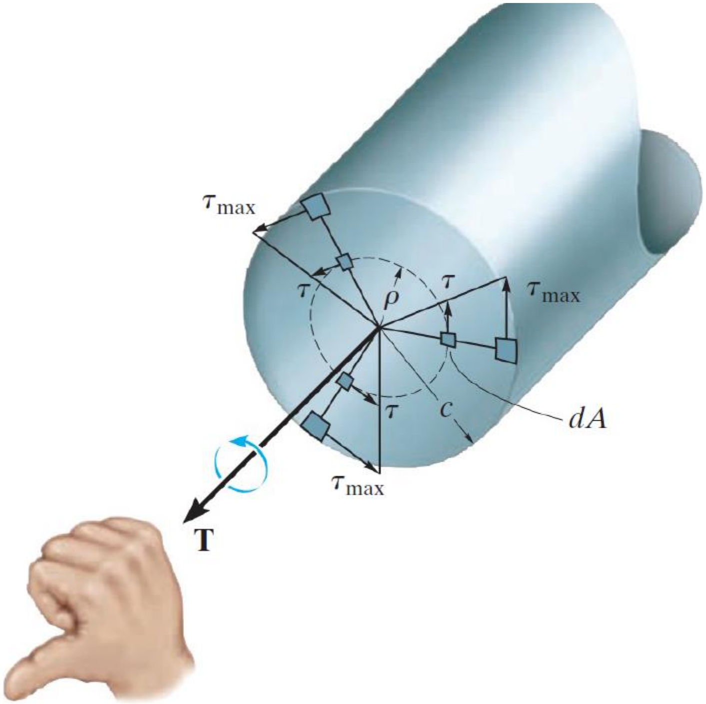
polar moment of inertia

THE TORSION FORMULA

$$T = \frac{\tau_{max}}{c} \int_A \rho^2 \, dA$$

$$\tau_{max} = \frac{Tc}{J}$$

$$\tau = \frac{T\rho}{J}$$



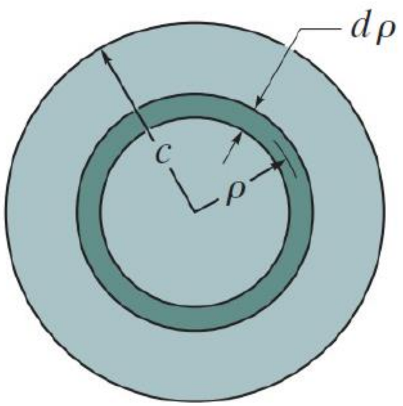
Shear stress varies linearly along each radial line of the cross section.

Polar Moment of Inertia.

$$\tau = \frac{T\rho}{J}$$

$$J = \int_A \rho^2 \, dA = \int_0^c \rho^2 (2\pi\rho \, d\rho)$$

$$= 2\pi \int_0^c \rho^3 \, d\rho = 2\pi \left(\frac{1}{4}\right) \rho^4 \Big|_0^c$$

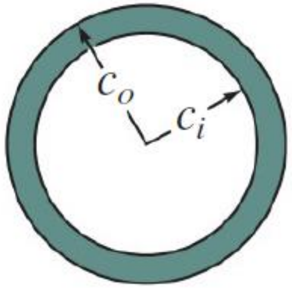


$$J = \frac{\pi}{2} c^4$$

Solid Section

Note that J is always positive. Common units used for its measurement are mm^4 or in^4.

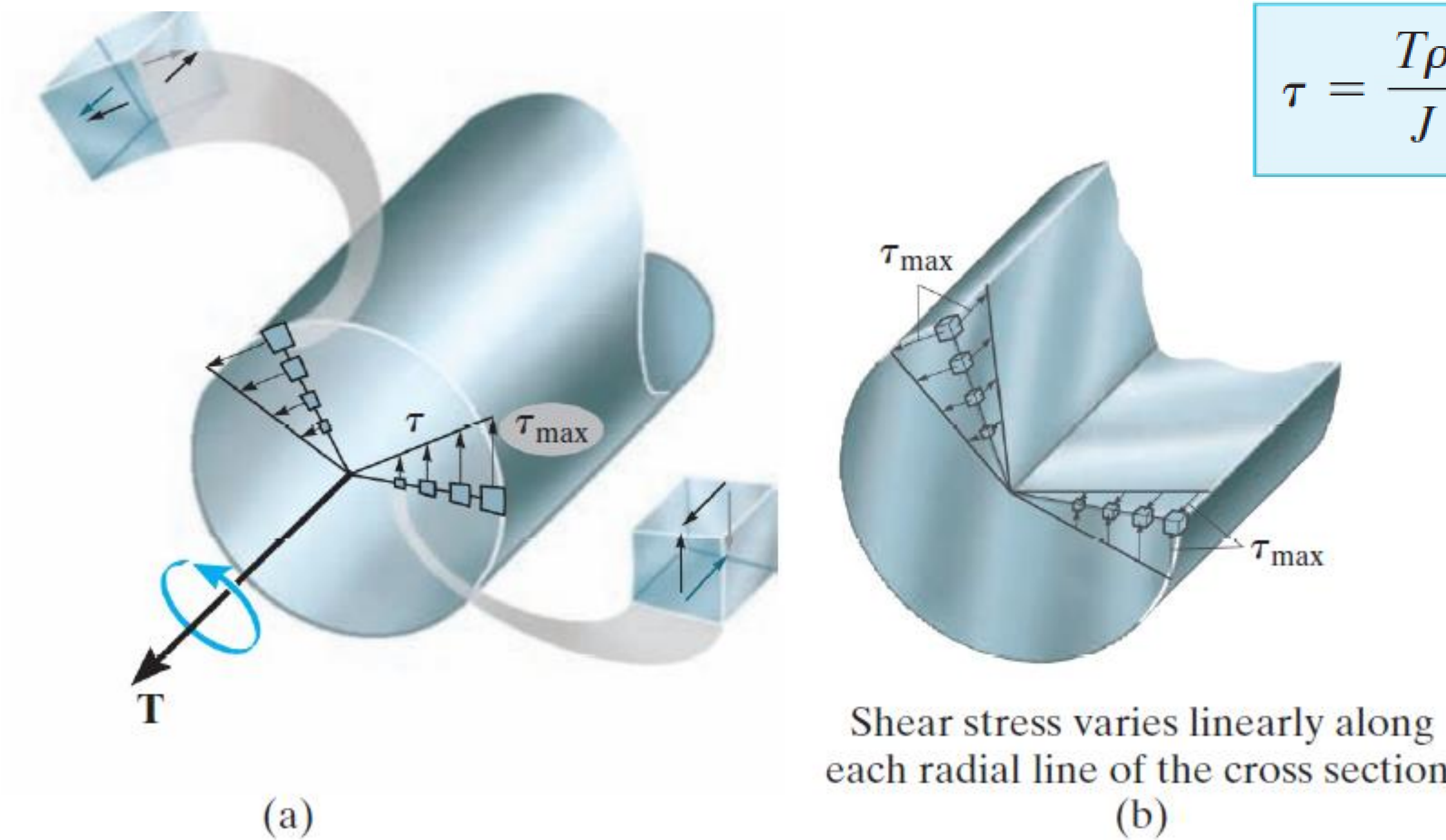
$$J = \frac{\pi}{2} (c_o^4 - c_i^4)$$



Tube

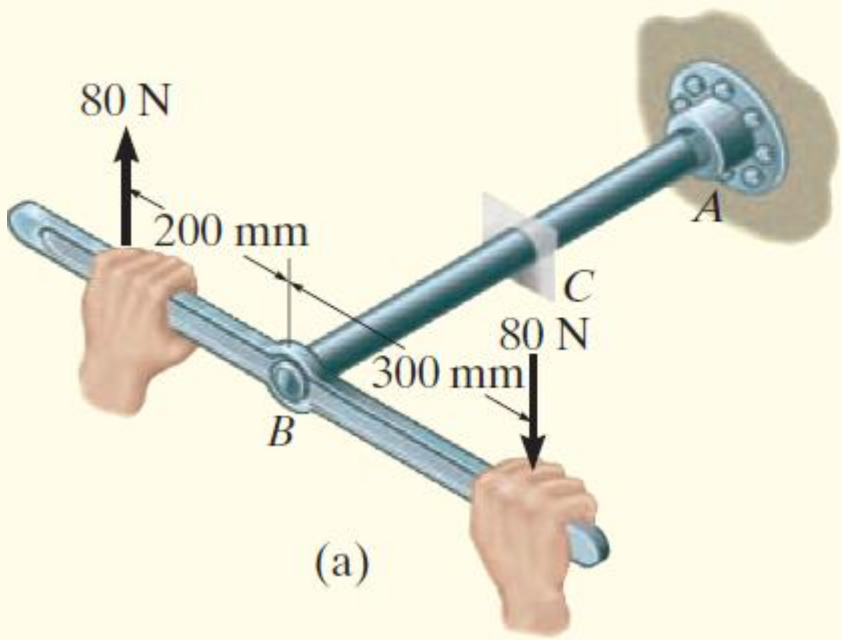
Shear Stress Distribution

$$\tau = \frac{T\rho}{J}$$



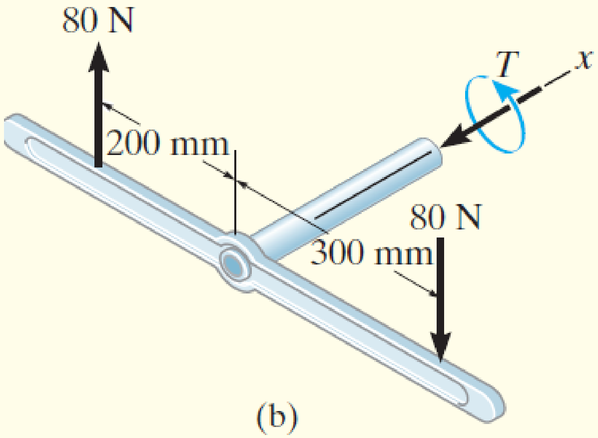
Shear stress varies linearly along each radial line of the cross section.

The pipe shown in Fig. 5–10*a* has an inner radius of 40 mm and an outer radius of 50 mm. If its end is tightened against the support at *A* using the torque wrench, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe.



(a)

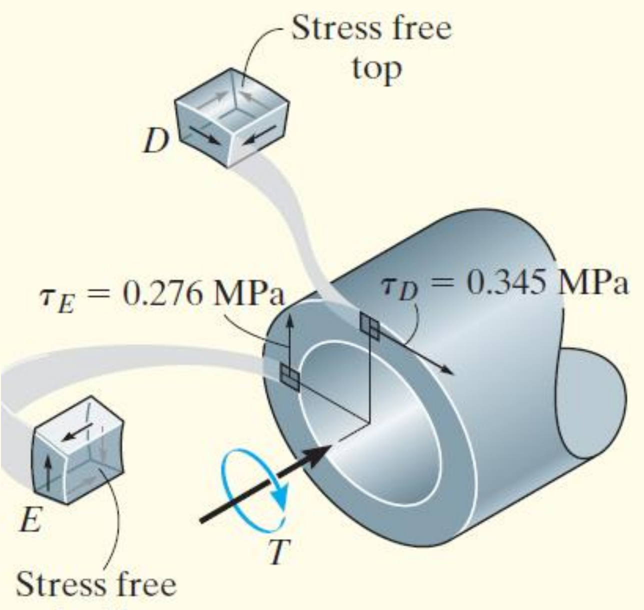
Internal Torque.



(b)

Section Property.

Shear Stress.



(c)

POWER TRANSMISSION

Power is defined as the work performed per unit of time.

$$P = \frac{T\,d\theta}{dt}$$

Since the shaft's angular velocity is $\omega = d\theta/dt$, then the power is

$$P = T\omega$$

$$P = T\omega$$

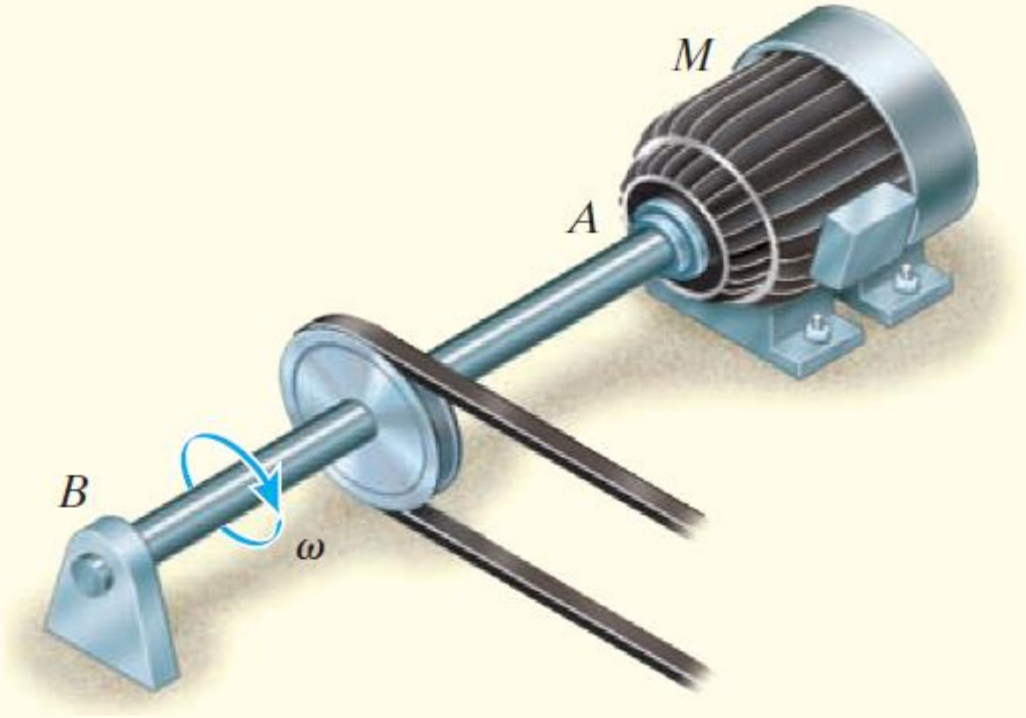
In the SI system, power is expressed in *watts* when torque is measured in newton-meters $(N \cdot m)$ and ω is in radians per second (rad/s) $(1\ W = 1\ N \cdot m/s)$. In the FPS system, the basic units of power are foot-pounds per second $(ft \cdot lb/s)$; however, *horsepower* (hp) is often used in engineering practice, where

$$1\ hp = 550\ ft \cdot lb/s$$

For machinery, the *frequency* of a shaft's rotation, f, is often reported. This is a measure of the number of revolutions or "cycles" the shaft makes per second and is expressed in hertz $(1\ Hz = 1\ cycle/s)$. Since 1 cycle $= 2\pi$ rad, then $\omega = 2\pi f$, and so the above equation for power can also be written as

$$P = 2\pi f T$$

A solid steel shaft AB, shown in Fig. 5–11, is to be used to transmit 5 hp from the motor M to which it is attached. If the shaft rotates at $\omega = 175$ rpm and the steel has an allowable shear stress of $\tau_{allow} = 14.5$ ksi, determine the required diameter of the shaft to the nearest $\frac{1}{8}$ in.



ANGLE OF TWIST

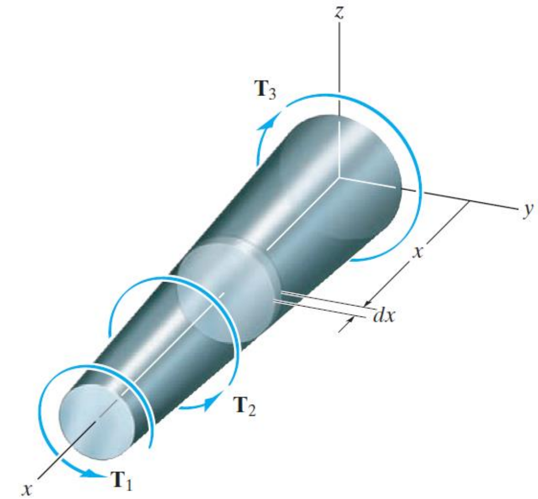
$$d\phi = \gamma \frac{dx}{\rho}, \quad \gamma = \frac{d\phi}{dx}\rho$$

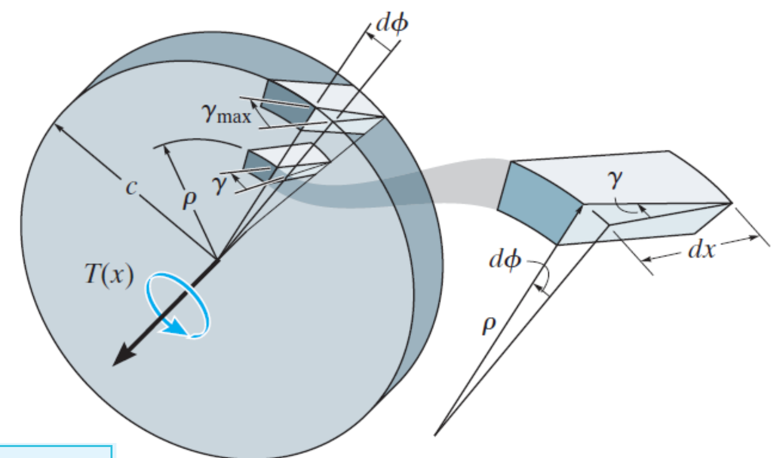
$$\tau = \gamma G, \quad \tau = \frac{T(x)\,\rho}{J(x)}$$

$$\downarrow$$

$$\frac{T(x)\,\rho}{J(x)} = \gamma G = \frac{d\phi}{dx}\rho G$$

$$\frac{T(x)}{J(x)} = \frac{d\phi}{dx} G$$



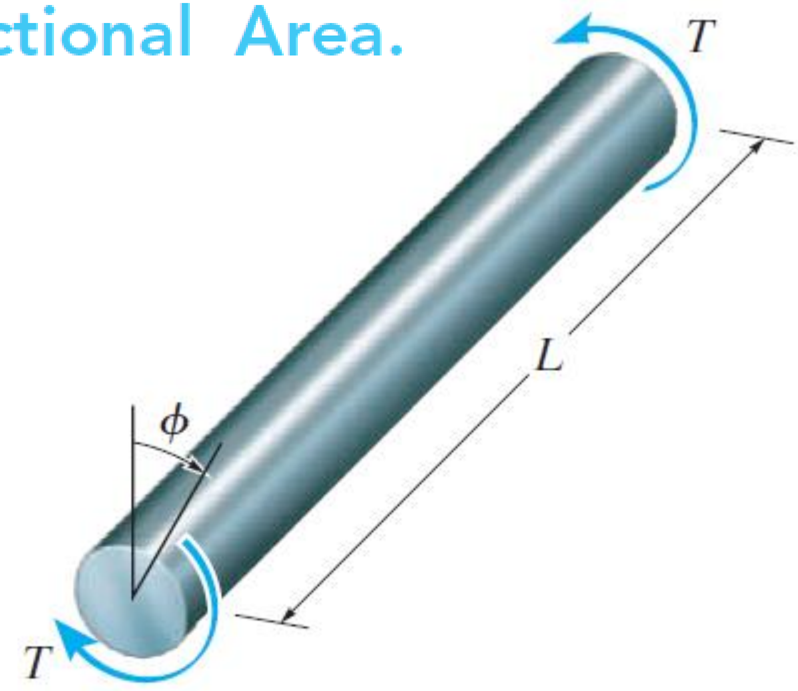


$$d\phi = \frac{T(x)}{J(x)G(x)}\,dx$$

$$\phi = \int_0^L \frac{T(x)\,dx}{J(x)G(x)}$$

ANGLE OF TWIST

Constant Torque and Cross-Sectional Area.

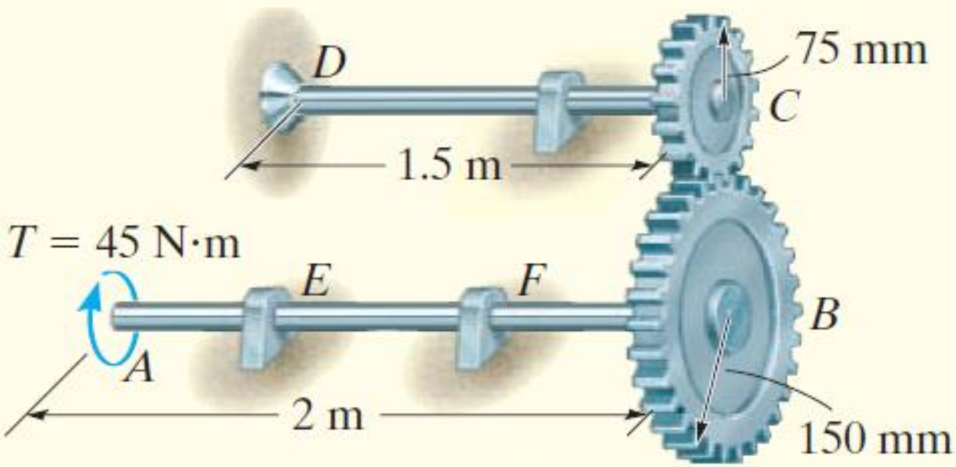


$$\phi = \frac{TL}{JG}$$

Multiple Torques.

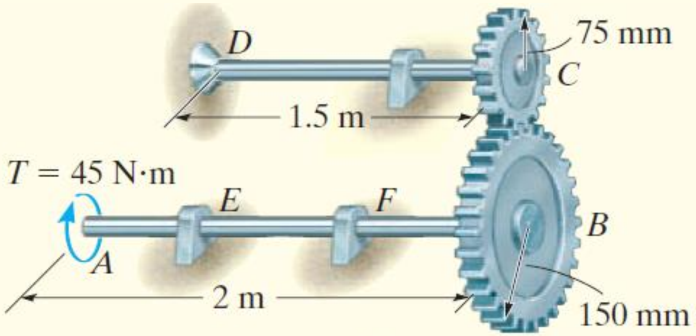
$$\phi = \sum \frac{TL}{JG}$$

The two solid steel shafts shown in Fig. 5–18*a* are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque $T = 45\ \text{N}\cdot\text{m}$ is applied. Shaft DC is fixed at D. Each shaft has a diameter of 20 mm. $G = 80$ GPa.



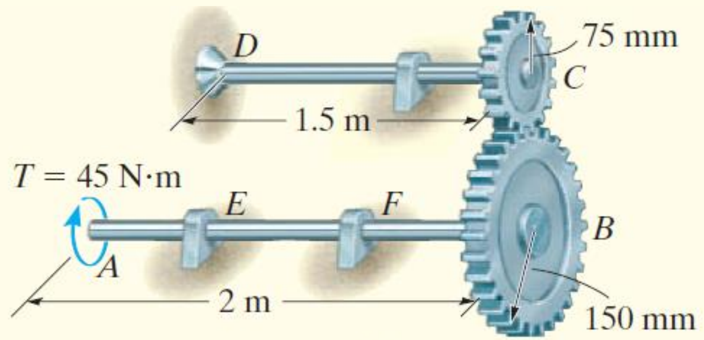
(a)

Internal Torque.

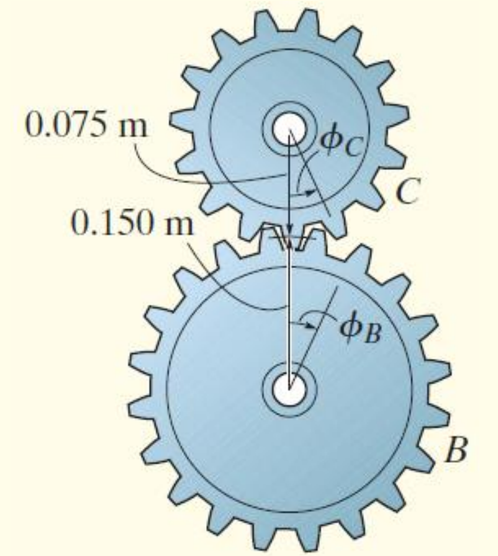


(a)

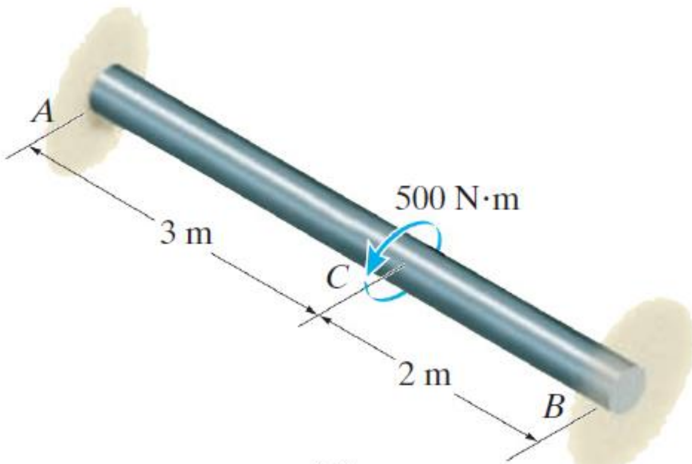
Angle of Twist.



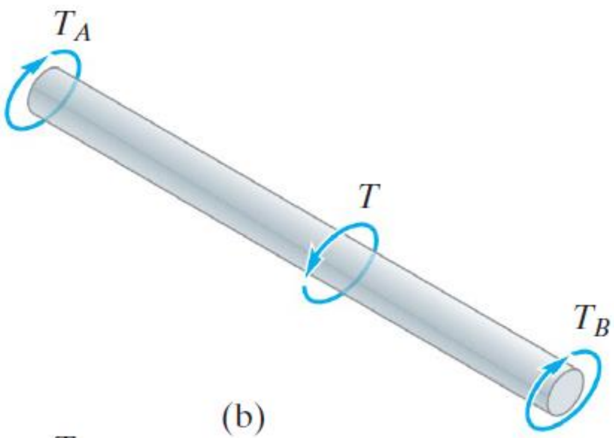
(a)



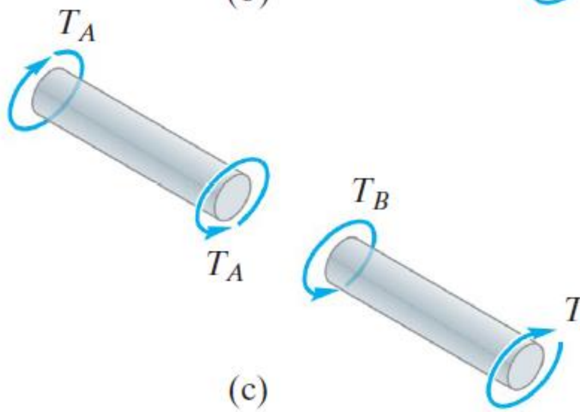
STATICALLY INDETERMINATE TORQUE-LOADED MEMBERS



(a)



(b)



(c)

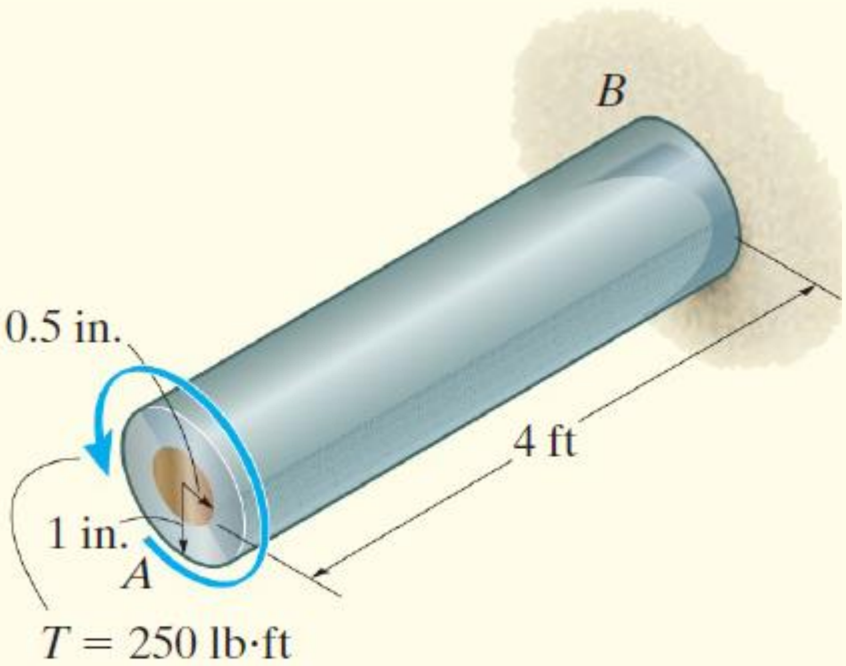
$$\Sigma M = 0;$$

$$500\ \text{N}\cdot\text{m} - T_A - T_B = 0$$

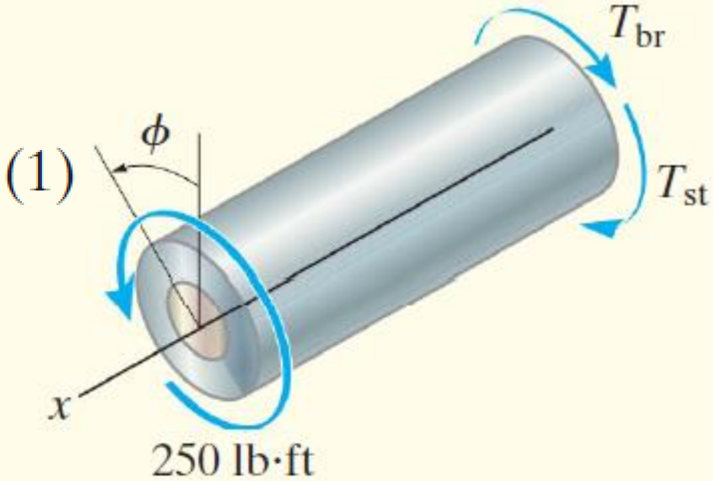
$$\phi_{A/B} = 0$$

$$\frac{T_A(3\ \text{m})}{JG} - \frac{T_B(2\ \text{m})}{JG} = 0 \qquad T_A = 200\ \text{N}\cdot\text{m} \quad \text{and} \quad T_B = 300\ \text{N}\cdot\text{m}$$

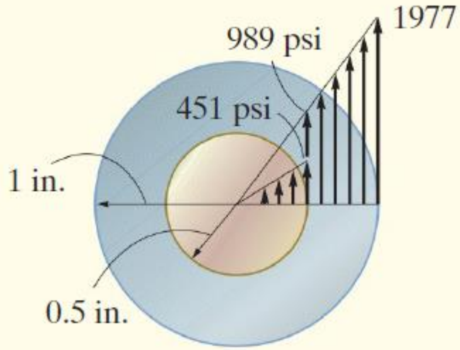
The shaft shown in Fig. 5–22*a* is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250\ \text{lb}\cdot\text{ft}$ is applied at its end, plot the shear-stress distribution along a radial line on its cross section. Take $G_{st} = 11.4(10^3)$ ksi, $G_{br} = 5.20(10^3)$ ksi.



Equilibrium.

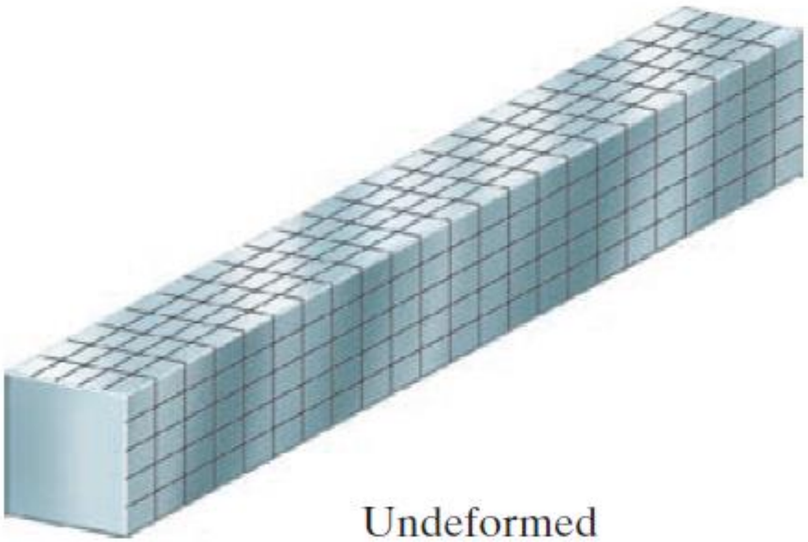


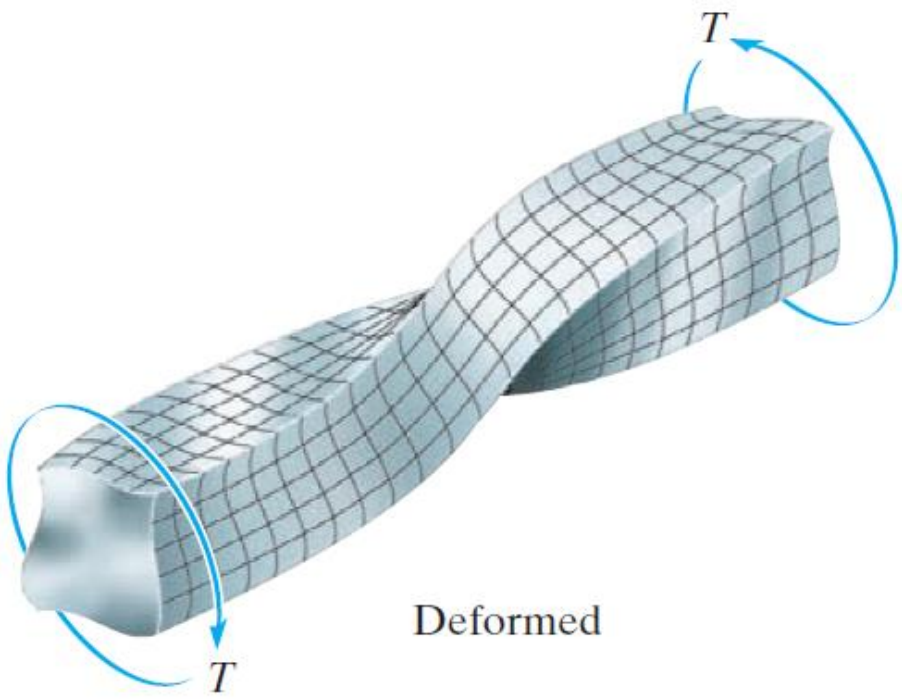
Compatibility.



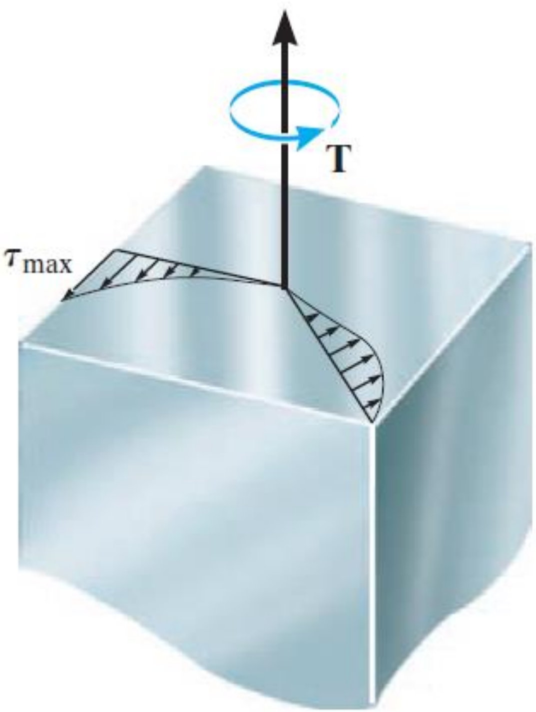
Shear–stress distribution

SOLID NONCIRCULAR SHAFTS

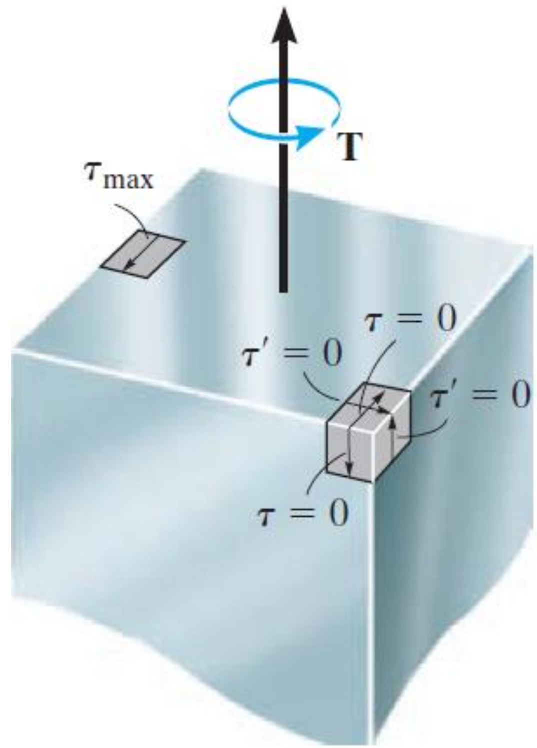




SOLID NONCIRCULAR SHAFTS

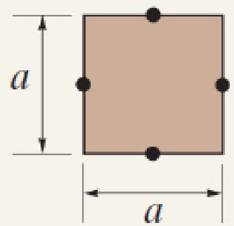
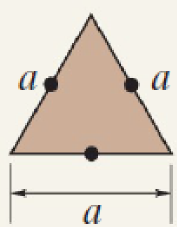
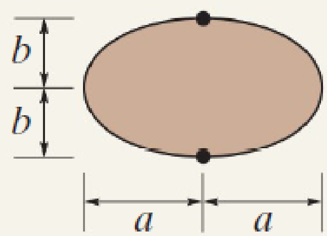


Shear stress distribution along two radial lines

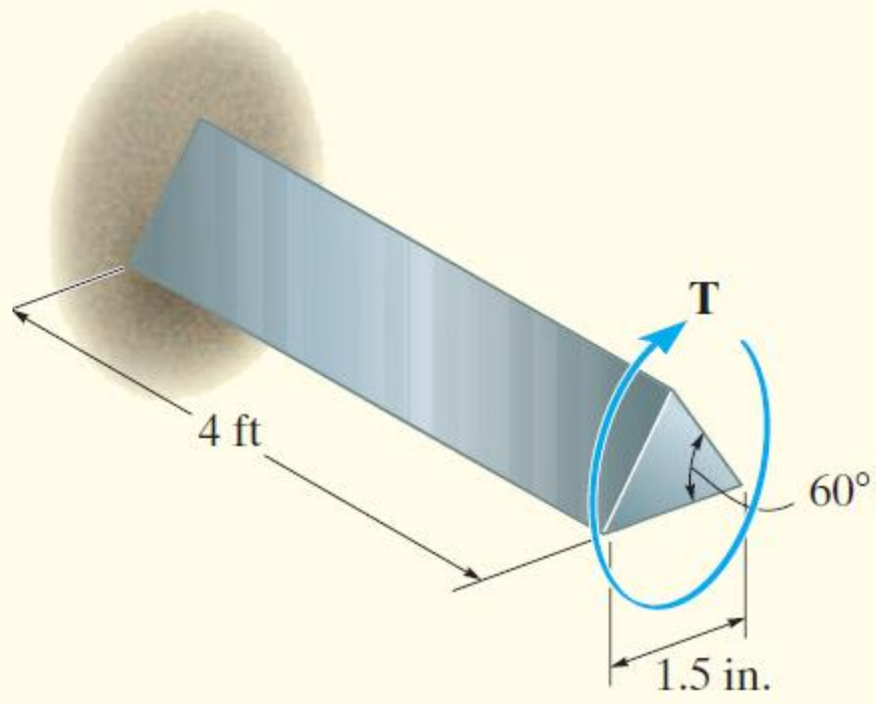


Warping of cross-sectional area

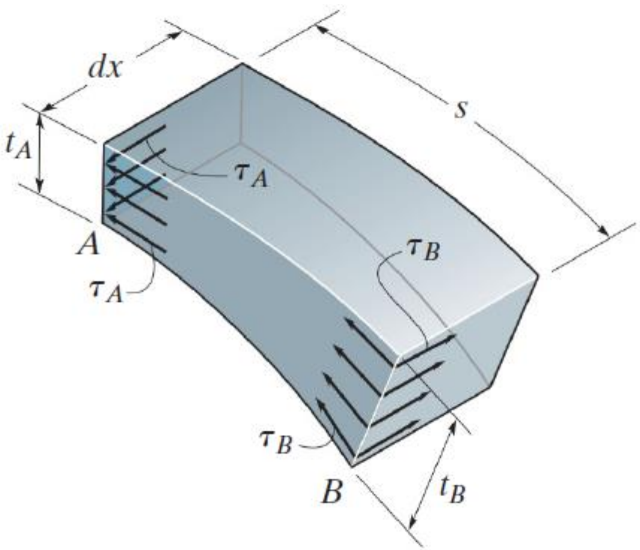
SOLID NONCIRCULAR SHAFTS

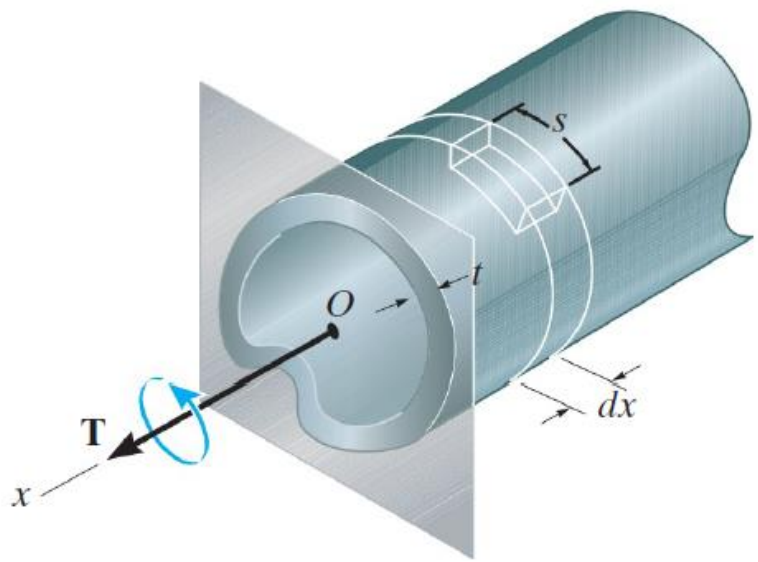
Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4G}$
Ellipse 	$\frac{2\,T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

The 6061-T6 aluminum shaft shown in Fig. 5–25 has a cross-sectional area in the shape of an equilateral triangle. Determine the largest torque **T** that can be applied to the end of the shaft if the allowable shear stress is $\tau_{allow} = 8$ ksi and the angle of twist at its end is restricted to $\phi_{allow} = 0.02$ rad. How much torque can be applied to a shaft of circular cross section made from the same amount of material?



THIN-WALLED TUBES HAVING CLOSED CROSS SECTIONS





$$dF_A = \tau_A (t_A \, dx)$$

$$dF_B = \tau_B \, (t_B \, dx)$$

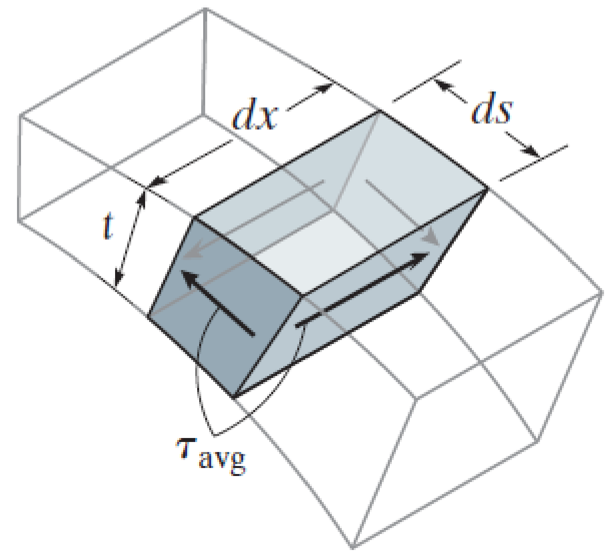
$$\tau_A \, t_A = \tau_B \, t_B$$

Shear Flow.

$$q = \tau_{\text{avg}} t$$

$$dA = t\,ds, \text{ so that } dF = \tau_{avg}(t\,ds) = q\,ds.$$

$$q = dF/ds.$$

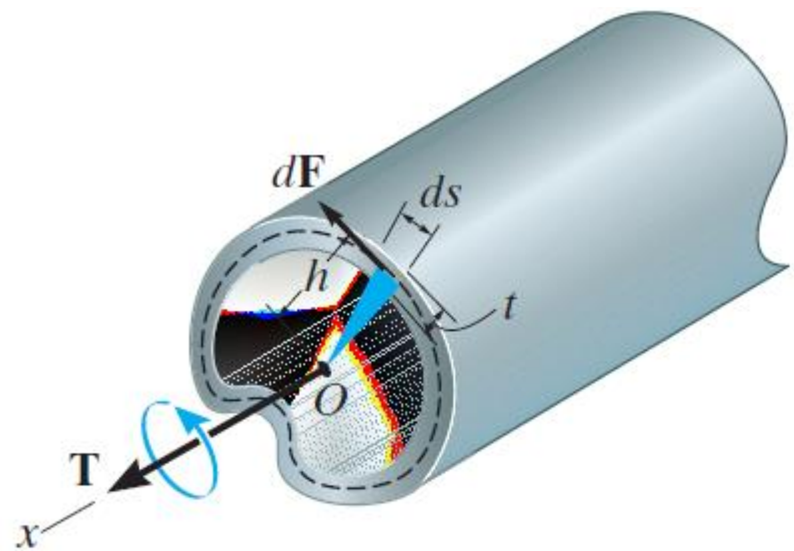


$$dF = \tau_{avg}\,dA = \tau_{avg}\,(t\,ds)$$

$$dT = h(dF) = h(\tau_{avg}\,t\,ds)$$

$$T = \oint h\tau_{avg}\,t\,ds$$

$$q = \tau_{avg}\,t \text{ is constant}$$



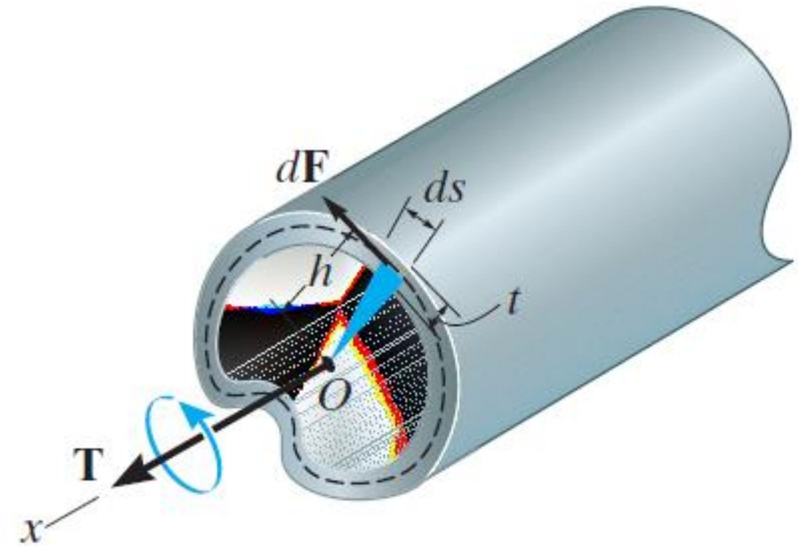
$$dF = \tau_{avg}\, dA = \tau_{avg}\,(t\, ds)$$

$$dT = h(dF) = h(\tau_{avg}\, t\, ds)$$

$$T = \oint h\tau_{avg}\, t\, ds$$

$$q = \tau_{avg}\, t \text{ is } \textit{constant}$$

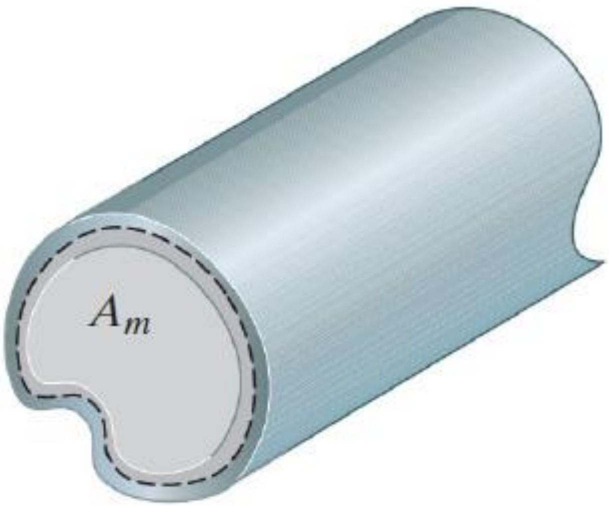
$$T = \tau_{avg} t \oint h\, ds$$



A graphical simplification can be made for evaluating the integral by noting that the *mean area*, shown by the blue colored triangle in Fig. 5–26*d*, is $dA_m = (1/2)h\, ds$. Thus,

$$T = 2\tau_{avg}\, t \int dA_m = 2\tau_{avg}\, tA_m$$

$$T = 2\tau_{avg} t \int dA_m = 2\tau_{avg} t A_m$$



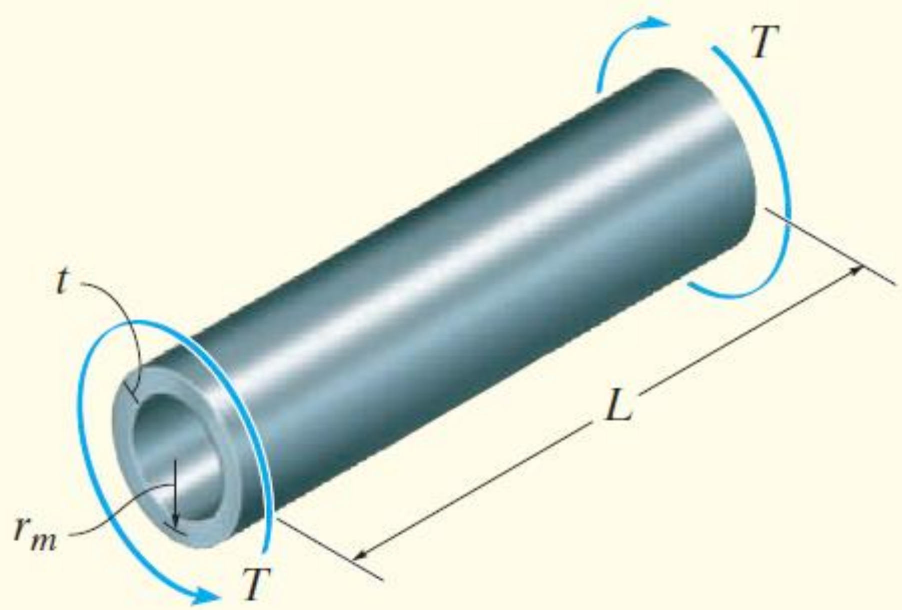
$$\tau_{avg} = \frac{T}{2tA_m}$$

$$q = \frac{T}{2A_m}$$

Angle of Twist. $$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

Here again the integration must be performed around the entire boundary of the tube's cross-sectional area.

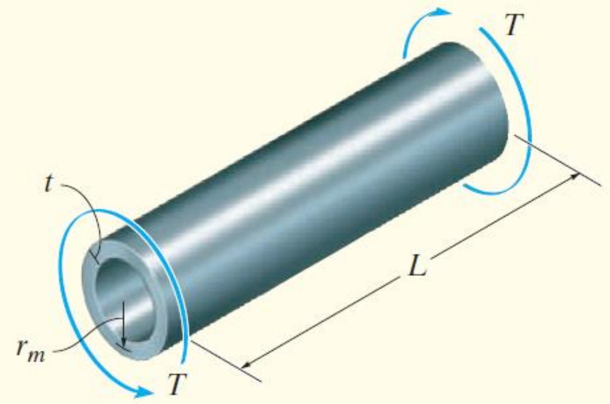
Calculate the average shear stress in a thin-walled tube having a circular cross section of mean radius r_m and thickness t, which is subjected to a torque T, Fig. 5–27a. Also, what is the relative angle of twist if the tube has a length L?



Average Shear Stress. The mean area for the tube is $A_m = \pi r_m^2$. Applying Eq. 5–18 gives

$$\tau_{avg} = \frac{T}{2tA_m} = \frac{T}{2\pi t r_m^2} \qquad Ans.$$

We can check the validity of this result by applying the torsion formula. Here



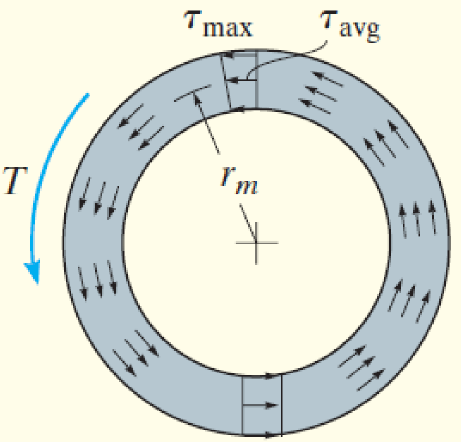
$$J = \frac{\pi}{2}(r_o^4 - r_i^4)$$

$$= \frac{\pi}{2}(r_o^2 + r_i^2)(r_o^2 - r_i^2)$$

$$= \frac{\pi}{2}(r_o^2 + r_i^2)(r_o + r_i)(r_o - r_i)$$

Since $r_m \approx r_o \approx r_i$ and $t = r_o - r_i$, $J = \frac{\pi}{2}(2r_m^2)(2r_m)t = 2\pi r_m^3 t$

$$\tau_{avg} = \frac{Tr_m}{J} = \frac{Tr_m}{2\pi r_m^3 t} = \frac{T}{2\pi t r_m^2} \qquad Ans.$$



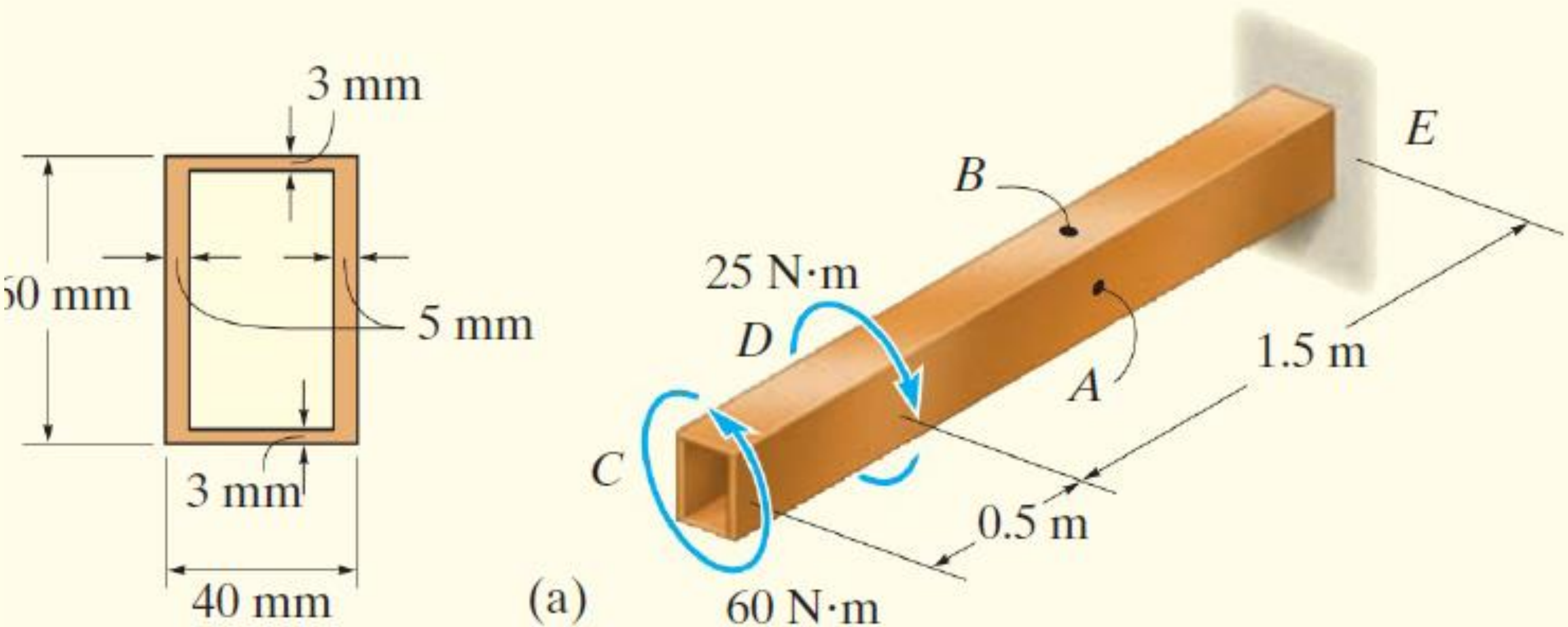
Angle of Twist. Applying Eq. 5–20, we have

$$\phi = \frac{TL}{4A_m^2G}\oint\frac{ds}{t} = \frac{TL}{4(\pi r_m^2)^2Gt}\oint ds$$

The integral represents the length around the centerline boundary, which is $2\pi r_m$. Substituting, the final result is

$$\phi = \frac{TL}{2\pi r_m^3 Gt} \qquad \textit{Ans.}$$

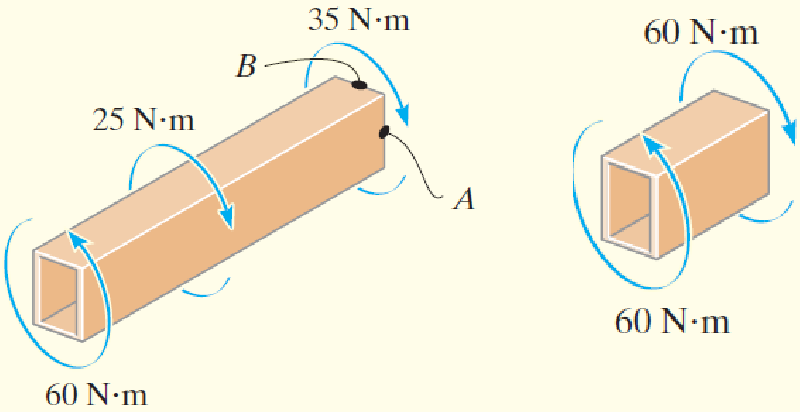
The tube is made of C86100 bronze and has a rectangular cross section as shown in Fig. 5–28*a*. If it is subjected to the two torques, determine the average shear stress in the tube at points *A* and *B*. Also, what is the angle of twist of end *C*? The tube is fixed at *E*.



(a)

Angle of Twist. From the free-body diagrams in Fig. 5–28*b* and 5–28*c*, the internal torques in regions *DE* and *CD* are 35 N · m and 60 N · m, respectively. Following the sign convention outlined in Sec. 5.4, these torques are both positive. Thus, Eq. 5–20 becomes

$$\phi = \sum \frac{TL}{4A_m^2\, G} \oint \frac{ds}{t}$$



THIN-WALLED TUBES HAVING OPENED CROSS SECTIONS

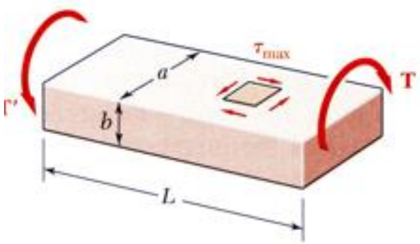


TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

- For uniform rectangular cross-sections,

$$\tau_{max} = \frac{T}{c_1 ab^2} \qquad \phi = \frac{TL}{c_2 ab^3 G}$$

- At large values of a/b, the maximum shear stress and angle of twist for other open sections are the same as a rectangular bar.