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Linear Programming

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Introduction

- Many problems take the form of maximizing or minimizing an objective, given limited resources and competing constraints.

- If we can specify the objective as a linear function of certain variables, and if we can specify the constraints on resources as equalities or inequalities on those variables, then we have a **linear programming problem**.

- Linear programs arise in a variety of practical applications

A political problem

- Suppose that you are a politician trying to win an election
- our district has three different types of areas—urban, suburban, and rural
- These areas have, respectively, 100,000, 200,000, and 50,000 registered voters.
- You would like at least half the registered voters in each of the three regions to vote for you.
- Your primary issues are building more roads, gun control, farm subsidies, and a gasoline tax dedicated to improved public transit.

A political problem (cont.)

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

- you can estimate how many votes you win or lose from each population segment by spending \$1,000 on advertising on each issue.
- each entry indicates the number of thousands of either urban, suburban, or rural voters who would be won over by spending \$1,000 on advertising in support of a particular issue.
- Negative entries denote votes that would be lost.
- Your task is to figure out the minimum amount of money that you need to spend in order to win 50,000 urban votes, 100,000 suburban votes, and 25,000 rural votes.

formulate this question mathematically

- $x1$ is the number of thousands of dollars spent on advertising on building roads,
- $x2$ is the number of thousands of dollars spent on advertising on gun control,
- $x3$ is the number of thousands of dollars spent on advertising on farm subsidies,
- and
- $x4$ is the number of thousands of dollars spent on advertising on a gasoline tax.

- We can write the requirement that we win at least 50,000 urban votes as

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50 .$$

- Similarly, we can write the requirements that we win at least 100,000 suburban votes and 25,000 rural votes as

$$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$$

and

$$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25 .$$

- In order to keep costs as small as possible, you would like to minimize the amount spent on advertising. That is, you want to minimize the expression

$$x_1 + x_2 + x_3 + x_4 \, .$$

- Although negative advertising often occurs in political campaigns, there is no such thing as negative-cost advertising. Consequently, we require that

$$x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0, \;\; \text{and} \;\; x_4 \geq 0 \, .$$

Linear Programming

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
& -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
& 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
& 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
& & & x_1, x_2, x_3, x_4 & & & & & \geq & 0\ .
\end{array}
$$

- The solution of this linear program yields your optimal strategy

General linear programs

- In the general linear-programming problem, we wish to optimize a linear function subject to a set of linear inequalities. Given a set of real numbers $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, we define a ***linear function*** f on those variables by

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{j=1}^{n} a_jx_j .$$

- If b is a real number and f is a linear function, then the equation

$$f(x_1, x_2, \ldots, x_n) = b$$

is a ***linear equality*** and the inequalities

$$f(x_1, x_2, \ldots, x_n) \le b$$

and

$$f(x_1, x_2, \ldots, x_n) \ge b$$

are ***linear inequalities***.

Minimization or Maximization Linear Program

- Formally, a ***linear-programming problem*** is the problem of either minimizing or maximizing a linear function subject to a finite set of linear constraints.

- If we are to minimize, then we call the linear program a ***minimization linear program***,

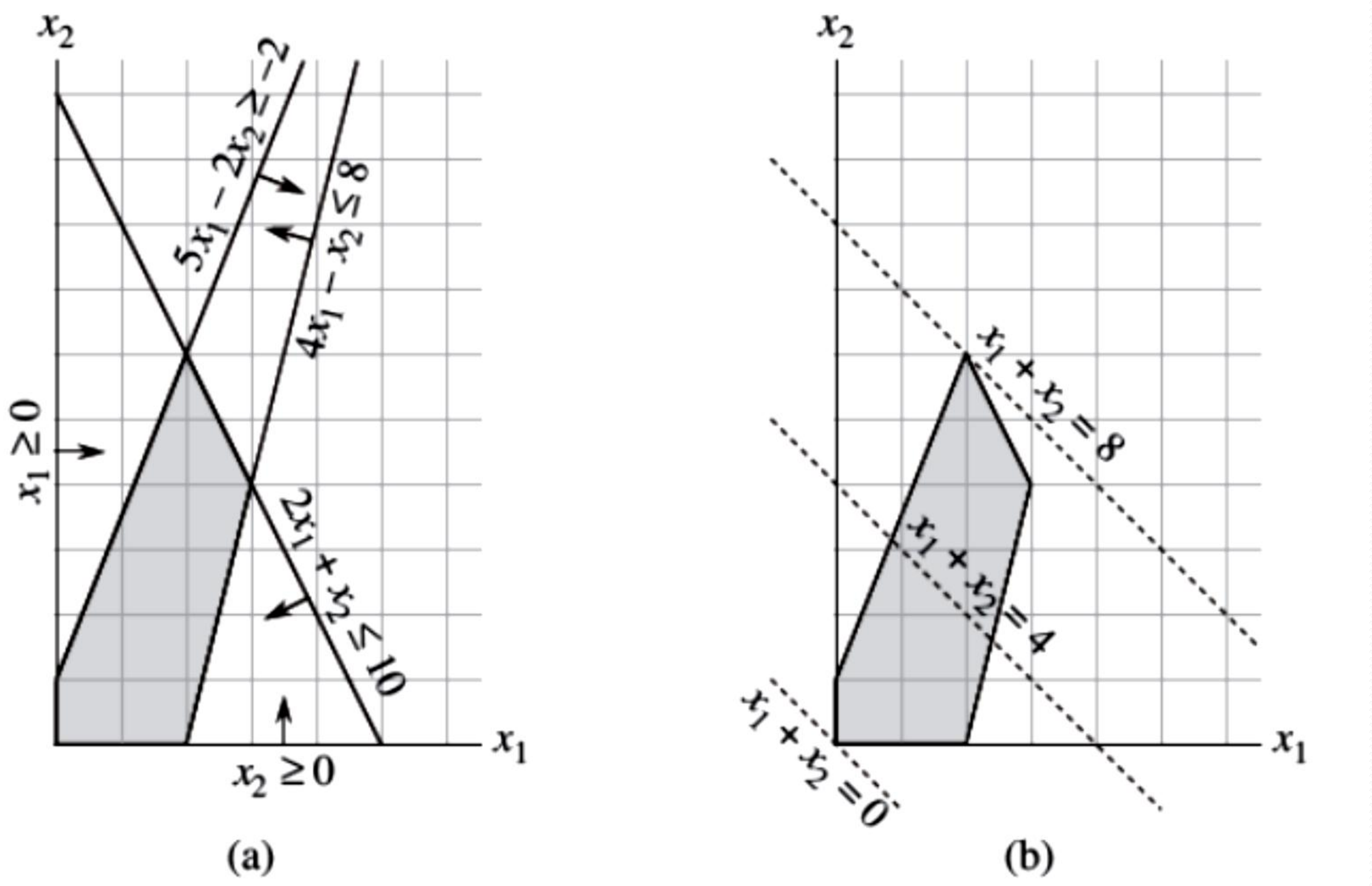
- and if we are to maximize, then we call the linear program a ***maximization linear program***.

- Informally, a linear program in standard form is the maximization of a linear function subject to linear *inequalities*, whereas a linear program in slack form is the maximization of a linear function subject to linear *equalities*.

Example

- We call any setting of the variables x_1 and x_2 that satisfies all the constraints a ***feasible solution*** to the linear program.

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \le & 8 \\
 & 2x_1 & + & x_2 & \le & 10 \\
 & 5x_1 & - & 2x_2 & \ge & -2 \\
 & & x_1, x_2 & & \ge & 0 .
\end{array}
$$



x_2
$5x_1 - 2x_2 \geq -2$
$4x_1 - x_2 \leq 8$
$x_1 \geq 0$
$2x_1 + x_2 \leq 10$
$x_2 \geq 0$
x_1
(a)
x_2
$x_1 + x_2 = 8$
$x_1 + x_2 = 4$
$x_1 + x_2 = 0$
x_1
(b)

simplex algorithm

- The ***simplex algorithm*** takes as input a linear program and returns an optimal solution.
- It starts at some vertex of the simplex and performs a sequence of iterations.
- In each iteration, it moves along an edge of the simplex from a current vertex to a neighboring vertex whose objective value is no smaller than that of the current vertex (and usually is larger.)
- The simplex algorithm terminates when it reaches a local maximum, which is a vertex from which all neighboring vertices have a smaller objective value.
- Because the feasible region is convex and the objective function is linear, this local optimum is actually a global optimum

Integer Linear Programming (ILP)

- If we add to a linear program the additional requirement that all variables take on integer values, we have an ***integer linear program***.

- just finding a feasible solution to this problem is NP-hard; since no polynomial-time algorithms are known for any NP-hard problems, there is no known polynomial-time algorithm for integer linear programming.

- In contrast, we can solve a general linear-programming problem in polynomial time.

Standard form

In ***standard form***, we are given n real numbers $c_1, c_2, \ldots, c_n$; m real numbers $b_1, b_2, \ldots, b_m$; and mn real numbers a_{ij} for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We wish to find n real numbers $x_1, x_2, \ldots, x_n$ that

$$
\begin{array}{lrcll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\
\text{subject to} & & & & \\
& \displaystyle\sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for } i = 1, 2, \ldots, m \\
& x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n \,.
\end{array}
$$

- Once we cast a problem as a polynomial-sized linear program, we can solve it in polynomial time by the ellipsoid algorithm or interior-point methods.

- Several linear-programming software packages can solve problems efficiently, so that once the problem is in the form of a linear program, such a package can solve it.

Formulating a problem – Let’s manufacture some chocolates

- Consider a chocolate manufacturing company which produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, following quantities are required:
- Each unit of A requires 1 unit of Milk and 3 units of Choco
- Each unit of B requires 1 unit of Milk and 2 units of Choco
- The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of
- Rs 6 per unit A sold
- Rs 5 per unit B sold.

Let's manufacture some chocolates

- Now, the company wishes to maximize its profit. How many units of A and B should it produce respectively?

	Milk	Choco	Profit per unit
A	1	3	Rs 6
B	1	2	Rs 5
Total	5	12	

Let's manufacture some chocolates

- Let the total number of units produced of A be = X
- Let the total number of units produced of B be = Y
- Now, the total profit is represented by Z

Let’s manufacture some chocolates

- **Profit: Max Z = 6X+5Y**
- **X+Y ≤ 5**
- **3X+2Y ≤ 12**
- **X ≥ 0 & Y ≥ 0**