Non-subnormal subgroups of groups
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According to Möhres’s Theorem an arbitrary group whose proper subgroups are all subnormal (or a group without non-subnormal proper subgroups) is solvable. In this paper we generalize Möhres’s Theorem, by proving that every group with at most 56 non-subnormal subgroups is solvable. Also we show that the derived length of a solvable group with a finite number \( k \) of non-\( n \)-subnormal subgroups is bounded in terms of \( n \) and \( k \).

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1. Introduction and results

Let \( G \) be a group. A subgroup \( H \) of \( G \) is said to be subnormal in \( G \) if there exists a finite series of subgroups of \( G \) such that

\[
H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_k = G.
\]

If \( H \) is subnormal in \( G \), then the defect of \( H \) in \( G \) is the shortest length of such a series. We shall say that a subgroup \( H \) of \( G \) is \( n \)-subnormal if \( H \) is subnormal of defect at most \( n \).

In a famous paper of 1965, Roseblade [7] proved that a group with all subgroups \( n \)-subnormal is nilpotent of class \( \mu(n) \), where \( \mu(n) \) is a function that depends only on \( n \) (we can take \( \mu(n) \) to be best possible for all \( n \)), and so it is solvable of class at most

\[
[\log_2(\mu(n))] + 1. \tag{\ast}
\]

However, \( \mu(n) \) is not explicitly given in [7] and the exact values are only known for \( n \in \{1, 2\} \). In fact, results of Heineken [1] and Mahdavianary [5] state that a group with all cyclic subgroups 2-subnormal is nilpotent of class not exceeding 3. As a corollary of this result, it follows that \( \mu(2) \leq 3 \). Moreover, it follows from [9] that this bound is sharp. (In the case where \( n = 1 \), \( G \) is a Dedekind group and so \( \mu(1) \leq 2 \).

In 1990 Möhres [6] proved that a group with all subgroups \( n \)-subnormal (or a group without non-subnormal subgroups) is solvable. In this paper we study groups with finitely many non-subnormal subgroups. Let \( k \) be a non-negative integer. We say that a group \( G \) is an \( \mathbb{R}(k) \)-group (\( \mathbb{M}(k) \)-group, resp.) if \( G \) has exactly \( k \) non-\( n \)-subnormal (non-subnormal, resp.) subgroups. We note that (for fixed \( k \) and \( n \)) \( \mathbb{R}(k) \) is contained in \( \mathbb{M}(l) \) for some \( l \) less than or equal to \( k \). To prove this, let \( n, k \) be fixed and \( G \) be in \( \mathbb{R}(k) \). Then \( G \) has exactly \( k \) non-\( n \)-subnormal subgroups, so certainly \( G \) has at most \( k \) non-\( n \)-subnormal subgroups, so \( G \) has at most \( k \) non-subnormal subgroups and so \( G \) has exactly \( l \) non-subnormal subgroups for some \( l \) less than or equal to \( k \) and so \( G \) lies in \( \mathbb{M}(l) \) for some \( l \) less than or equal to \( k \).

Clearly the \( \mathbb{R}(k) \)-groups \( G \) with \( k = 0 \) are those in which all subgroups are \( n \)-subnormal, and the \( \mathbb{M}(k) \)-groups \( G \) with \( k = 0 \) are those with all subgroups subnormal. We note that if \( G \) is an \( \mathbb{R}(k) \)-group (or \( \mathbb{M}(k) \)-group), then \( k \neq 1 \). This is because every conjugate of a non-\( n \)-subnormal subgroup is also a non-\( n \)-subnormal subgroup. Therefore in considering \( \mathbb{R}(k) \)-groups (or \( \mathbb{M}(k) \)-groups) we may assume that \( k \geq 2 \).

According to (\ast), the derived length of a solvable \( \mathbb{R}(0) \)-group is \( \leq [\log_2(\mu(n))] + 1 \). Here we obtain a result which is of independent interest, namely, that the derived length of solvable \( \mathbb{R}(k) \)-groups is bounded in terms of \( n \) and \( k \) (\( 2 \leq k \)).

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Theorem A. Let $G$ be a solvable $\mathfrak{R}_n(k)$-group (not necessarily finite) and $d$ be the derived length of $G$. Then $d \leq [\log_2(\mu(n))] + k$, where $\mu(n)$ is the function of Roseblade’s Theorem.

Also, we generalize M"ohres’s Theorem as follows (note that M"ohres’s Theorem says that every $\mathfrak{M}(0)$-group is solvable).

Theorem B. Suppose that $G$ is an $\mathfrak{M}(k)$-group ($\mathfrak{R}_n(k)$-group) for some $n$ with $k \leq 56$. Then $G$ is solvable. The alternating group of degree 5, $A_5$, is an $\mathfrak{M}(57)$-group.

2. Proofs

If $G$ is an arbitrary group, the norm $B_1(G)$ of $G$ is the intersection of all the normalizers of subgroups of $G$. This concept was introduced by R. Baer, and it is well-known [4, 8] that $Z(G) \leq B_1(G) \leq Z_2(G)$. Now we define $B_n(G)$ as the intersection of all the normalizers of non-$n$-subnormal subgroups of $G$, i.e.,

$$B_n(G) = \bigcap_{H \in \mathfrak{R}_n(G)} N_G(H),$$

where $\mathfrak{R}_n(G) = \{ H \mid H \text{ is a non-$n$-subnormal subgroup of } G \}$ (with the stipulation that $B_n(G) = G$ if all subgroups of $G$ are $n$-subnormal). Clearly $B_1(G) \leq B_2(G) \leq B_3(G) \leq \cdots$. Moreover, in view of the proof of Theorem A, below, we can see that

$$B_n(G) = \text{a nilpotent normal subgroup of } G \text{ of class } \leq \mu(n),$$

where $\mu(n)$ is the function of Roseblade’s Theorem. In fact $B_n(G)$ is a substantial generalization of the norm $B_1(G)$.

Proof of Theorem A. The group $G$ acts on the set

$$\mathfrak{R}_n(G) = \{ H \mid H \text{ is a non-$n$-subnormal subgroup of } G \}$$

by conjugation. By assumption, $|\mathfrak{R}_n(G)| = k$ (note that $k \geq 2$). Now the subgroup $B_n(G)$ is the kernel of this action, so $B_n(G)$ is normal in $G$ and $G/B_n(G) \twoheadrightarrow \text{Sym}_k$. It is surely well-known that the derived length of every solvable subgroup of the symmetric group $S_n$ of degree $n$ ($n \geq 1$) is at most $n - 1$. Hence $G/B_n(G)$ has derived length $\leq k - 1$. Therefore to complete the proof it is enough to show that $B_n(G)$ is solvable of class at most $[\log_2(\mu(n))] + 1$. To see this, according to the main result in [7], it is enough to show that every subgroup of $B_n(G)$ is $n$-subnormal. Suppose on the contrary that there exists a non-$n$-subnormal, say $H$, of $B_n(G)$. It follows that $H$ is a non-$n$-subnormal of $G$ and so, by definition of $B_n(G)$, we obtain that $H \not\leq B_n(G)$, which is impossible. Hence $B_n(G)$ is a nilpotent group of class at most $\mu(n)$ and so it is solvable of class at most $[\log_2(\mu(n))] + 1$. This completes the proof. $\square$

Combining the results quoted in the introduction and the above theorem, we obtain a nice corollary as follows:

Corollary 2.1. Let $G$ be a solvable $\mathfrak{R}_2(k)$-group, $d$ the derived length of $G$ and $k \geq 2$. Then $d \leq k$. 1.

Remark 2.2. In view of the proof of Theorem A, we can see that if $G$ is an arbitrary group with a finite number $k$ of non-$n$-subnormal subgroups, then the factor group $G/B_n(G)$ is finite and

$$\left| \frac{G}{B_n(G)} \right| \leq k!.$$ 

This result suggests that the behaviour of non-$n$-subnormal subgroups has a strong influence on the structure of the group.

Remark 2.3. The Wielandt subgroup $W(G)$ of a group $G$ is defined to be the intersection of all the normalizers of subnormal subgroups of $G$; this concept is naturally analogous to the norm of a group. Also we shall denote by $W'(G)$ the intersection of all the normalizers of non-subnormal subgroups of $G$. Now by an argument similar to that in the proof of Theorem A, mentioned for $B_n(G)$, we can see that every subgroup of $W(G)$ is subnormal and so, by M"ohres’s Theorem,

$$W'(G)$$

is a solvable normal subgroup of $G$.

Moreover, if $G$ is an arbitrary group with finitely many, $k$, non-subnormal subgroups, then the factor group $G/W'(G)$ is finite and

$$\left| \frac{G}{W'(G)} \right| \leq k!.$$ 

In the sequel, we want to prove Theorem B.

Lemma 2.4. Let $G$ be an $\mathfrak{M}(t)$-group and $H \leq G$; then $H$ is an $\mathfrak{M}(s)$-group for some $s \leq n$.

Proof. It is straightforward. $\square$

Lemma 2.5. Let $G$ be an $\mathfrak{M}(t)$-group, $K$ a normal subgroup of $G$, $\frac{G}{K} \in \mathfrak{M}(n)$ and $K \in \mathfrak{M}(m)$. Then $t \geq m + n$. 


Proof. For the proof it is enough to note that if $\frac{L}{H}$ is non-subnormal, then $H$ is non-subnormal. □

For any prime power $q$, we denote by $L_q(q)$ and $Sz(q)$, respectively, the projective special linear group of degree $n$ over the finite field of size $q$ and the Suzuki group over the field with $q$ elements. If $G$ is a finite group, then for every prime divisor $p$ of $|G|$, we denote by $v_p(G)$ the number of Sylow $p$-subgroups of $G$.

Proof of Theorem B. Suppose that $G$ is an $\mathfrak{M}(k)$-group with $k \leq 56$. Then in view of the proof of Theorem A and Remark 2.3, we get $\frac{G}{W(G)} \hookrightarrow \text{Sym}_k$. Therefore $\frac{G}{W(G)}$ is finite and also, by Lemma 2.5, $\frac{G}{W(G)}$ is an $\mathfrak{M}(r)$-group with $r < 57$. Thus replacing $G$ by the factor group $\frac{G}{W(G)}$, it can be assumed without loss of generality that $G$ is a finite $\mathfrak{M}(k)$-group with $k < 57$ (note that $W(G)$ is a solvable normal subgroup of $G$). Now suppose on the contrary that there exists a non-abelian finite insolvable $\mathfrak{M}(k)$-group of the least possible order, where $k < 57$. If there exists a non-trivial proper normal subgroup $N$ of $G$, then (by Lemmas 2.5 and 2.4) the groups $\frac{G}{N}$ and $N$ are in $\mathfrak{M}(s)$ with $s < 57$, and so they are solvable. It follows that $G$ is solvable, which is a contradiction. Therefore $G$ is a minimal simple $\mathfrak{M}(k)$-group with $k < 57$. By Thompson’s classification of minimal simple groups [10], $G$ is isomorphic to one of the following simple groups: $Alt_5 \cong L_2(5)$; $L_2(2m)$, $m$ an odd prime; $L_2(3m)$, $m$ an odd prime; $L_2(p)$, where $5 < p$ is prime and $p \equiv 2, 3 \pmod{5}$; $L_3(3)$ and $Sz(2^m)$, $m$ an odd prime. Now as $G$ is a simple group, every non-trivial proper subgroup (such as a Sylow $p$-subgroup or a cyclic subgroup) of $G$ is non-subnormal. But it is easy to see that all simple groups mentioned above have more than 57 non-trivial proper subgroups. In fact, if $G$ is isomorphic to $L_2(q)$, where $q = 5$ or 7, then it is easy to see (from the list of subgroups of $Alt_5$ and $L_2(7)$) that $Alt_5$ is an $\mathfrak{M}(57)$-group and $L_2(7)$ is an $\mathfrak{M}(177)$-group. If $G$ is isomorphic to $L_2(q)$, where $q = t^m \geq 8$, then as the group $L_2(t^m)$ has a partition $P$ consisting of $t^m + 1$ Sylow $t$-subgroups, $\frac{t^m + 1}{2}$ cyclic subgroups of order $\frac{t^{m-1}}{\gcd(2, t-1)}$ and $\frac{t^m - 1}{2}$ cyclic subgroups of order $\frac{t^m + 1}{\gcd(2, t+1)}$ (see pp. 185–187 and p. 193 of [3]), we obtain that $k \geq t^m + 1 + \frac{t^m + 1}{2} + \frac{t^m - 1}{2} \geq 9 + 36 + 28 = 73$. If $G \cong SL_3(3)$, then $|G| = 2^4 \times 3^3 \times 13$ and so $v_{13}(G) = 144 > 57$. If $G \cong Sz(q)$, then $|G| = q^2(q^2 + 1)(q - 1)$ and $v_2(G) = q^2 + 1 > 57$ (see Theorem 3.10 (and its proof) in Chapter XI of [2]). Finally, as mentioned above, $Alt_5$ is an $\mathfrak{M}(57)$-group and this completes the proof. □

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References